ON PROJECTIONS OF $L^\infty(G)$
ONTO TRANSLATION-INvariant SUBSPACES

BY

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1. Introduction. Let $G$ be a locally compact Abelian group with the
Haar measure $\mu$. Let $\Phi$ be a translation-invariant $*$-weakly closed subspace
of $L^\infty(G)$. Regard $L^\infty(G)$ and $\Phi$ as Banach spaces with the norm topology
of $L^\infty(G)$. The present note is a contribution to solution of the following
problem (P 1042): when is $\Phi$ complemented in $L^\infty(G)$? Let $\hat{G}$ be the dual
group of $G$. Denote by $\sigma(\Phi)$ the spectrum of $\Phi$, i.e., the set $\{\chi \in \hat{G}: \chi \in \Phi\}.
If $\Phi \neq \{0\}$, then $\sigma(\Phi) \neq \emptyset$. The main result of this note is Theorem 2
which gives a sufficient condition for $\Phi$ to be uncomplemented in $L^\infty(G)$
expressed in terms of $\sigma(\Phi)$.

2. Some examples. The first example of a complemented subspace
is the most general. Let $\Phi$ have a finite spectrum. Then $\Phi$ is the $*$-weak
closure of the linear space spanned by $\sigma(\Phi)$. Hence $\Phi$ is finite-dimensional and complemented in $L^\infty(G)$.

The second example involves a group $G$ of a special type. Let $G
= G_1 \oplus G_2$, where $G_1$ and $G_2$ are locally compact Abelian groups. Let $\mu_1
be the Haar measure on $G_1$. Let

$$ I = \left\{ f \in L^1(G): \int\limits_{G_1} f(s, t) \mu_1(ds) = 0 \text{ for almost all } t \in G_2 \right\}. $$

$I$ is a closed ideal in the group algebra $L^1(G)$, i.e., a translation-invariant closed subspace of $L^1(G)$. Let $g \in L^1(G_1)$ and

$$ \int\limits_{G_1} g \mu_1 = 1. $$

Write

$$ Pf(s, t) = f(s, t) - g(s) \int\limits_{G_1} f(u, t) \mu_1(du), \quad s \in G_1, t \in G_2. $$

$P$ is a continuous projection of $L^1(G)$ onto $I$. Denote by $\text{Ann}I$ the
annihilator of $I$, i.e., the set

$$ \{ \varphi \in L^\infty(G): \langle f, \varphi \rangle = 0 \text{ for every } f \in I \}, $$
where

$$\langle f, \varphi \rangle = \int_G f \varphi \, d\mu, \quad f \in L^1(G), \ \varphi \in L^\infty(G).$$

An $I$ is a translation-invariant \*-weakly closed subspace of $L^\infty(G)$, and $Q = E - P'$ is a continuous projection of $L^\infty(G)$ onto $\text{An} \ I$, where $E$ denotes the identity operator on $L^\infty(G)$. The spectrum of $\text{An} \ I$ may be identified with $\mathcal{G}_2$. In particular, if $\mathcal{G}_2$ is infinite, then $\text{An} \ I$ provides an example of a translation-invariant \*-weakly closed subspace of $L^\infty(G)$ which is complemented in $L^\infty(G)$ and has an infinite spectrum.

3. Main results. We introduce the following notation:

- $C_0(G) = \{\text{all continuous functions on } G \text{ with compact support}\}$,
- $C'_0(G) = \{\text{all continuous functions on } G \text{ vanishing at infinity}\}$,
- $C_u(G) = \{\text{all bounded uniformly continuous functions on } G\}$,
- $B(G) = \{\text{all bounded complex functions on } G\}$,
- $M(G) = \{\text{all bounded regular Borel measures on the field of Borel subsets of } G\}$.

**Theorem 1.** Let $\Phi$ be a translation-invariant \*-weakly closed subspace of $L^\infty(G)$ which is complemented in $L^\infty(G)$. Let $G$ be connected. Then $\Phi = L^\infty(G)$ or $\Phi \cap C_0(G) = \{0\}$.

**Proof.** We use argumentation based on ideas which go back to K. De Leeuw and are contained in [1], Theorem 4.1.

Suppose that $Q$ is a continuous projection of $L^\infty(G)$ onto $\Phi$. At the beginning we prove that there exists a continuous projection $R$ of $L^\infty(G)$ onto $\Phi$ such that

$$T_s R = R T_s \quad (1)$$

for every $s \in G$, where $T_s h(x) = h(x + s)$, $x \in G$, $h$ — any function on $G$.

Let $\mathcal{M}$ denote a Banach mean, i.e., a linear functional on $B(G)$ satisfying the following conditions:

- (i) $|\mathcal{M} \varphi| \leq \|\varphi\|$, $\|\varphi\| = \sup_{s \in G} |\varphi(s)|$,
- (ii) $\mathcal{M} T_s \varphi = \mathcal{M} \varphi$ for every $s \in G$,
- (iii) $\mathcal{M} c = c$ for any function $c$ constant on $G$.

The proof of the existence of a Banach mean may be found in [2], Theorem 1.2.1, p. 5.

Consider the function

$$\psi(f, \varphi)(s) = \langle f, T_{-s} Q T_s \varphi \rangle, \quad f \in L^1(G), \ \varphi \in L^\infty(G), \ s \in G.$$
For an arbitrarily fixed \( s \in G \) we have
\[
|\psi(f, \varphi)(s)| \leq \|f\|_1 \|T_{-s} QT_s \varphi\|_\infty \leq \|Q\| \|f\|_1 \|\varphi\|_\infty.
\]

Thus \( \psi(f, \varphi) \in B(G) \) and by (i) we have
\[\mathcal{M} \psi(f, \varphi) = \|Q\| \|f\|_1 \|\varphi\|_\infty,\]
whence the mapping \( f \to \mathcal{M} \psi(f, \varphi) \) with fixed \( \varphi \) is a linear continuous functional on \( L^1(G) \). It is represented in the form
\[\mathcal{M} \psi(f, \varphi) = \langle f, R\varphi \rangle\]
for some \( R\varphi \in L^\infty(G) \). From (3) and (2) it follows that \( R \) is a linear continuous operator and \( \|R\| \leq \|Q\| \). Let \( \Lambda_\Phi \Phi \) denote the annihilator of \( \Phi \), i.e., the set
\[\{f \in L^1(G) \colon \langle f, \varphi \rangle = 0 \text{ for every } \varphi \in \Phi\}.
\]

If \( \varphi \in L^\infty(G) \) and \( f \in \Lambda_\Phi \Phi \), then \( \psi(f, \varphi) = 0 \) and, by (3), \( R\varphi \in \Lambda_\Phi \Lambda_\Phi \Phi = \Phi \). If \( \varphi \in \Phi \), then \( T_{-s} QT_s \varphi = \varphi \), and by (iii) and (3) we obtain \( R\varphi = \varphi \).

Actually, we have proved that \( R \) is a continuous projection of \( L^\infty(G) \) onto \( \Phi \). In order to prove (1) notice that for every \( t \in G \) we have
\[
\psi(f, T_s \varphi)(t) = \langle f, T_{-t} QT_t T_s \varphi \rangle = \langle f, T_s T_{-(t+s)} QT_{t+s} \varphi \rangle \\
= \langle T_{-s} f, T_{-(t+s)} QT_{t+s} \varphi \rangle = \psi(T_{-s} f, \varphi)(t + s) \\
= T_{s} \psi(T_{-s} f, \varphi)(t).
\]

and (1) now follows from (ii), namely
\[
\langle f, RT_s \varphi \rangle = \mathcal{M} \psi(f, T_s \varphi) = \mathcal{M} T_s \psi(T_{-s} f, \varphi) = \mathcal{M} \psi(T_{-s} f, \varphi) \\
= \langle T_{-s} f, R\varphi \rangle = \langle f, T_s R\varphi \rangle.
\]

Now we show that \( C_u(G) \) is an invariant subspace of \( R \). For \( \varphi \in C_u(G) \) and an arbitrary \( \varepsilon > 0 \) there exists a symmetric neighbourhood of zero \( V_\varepsilon \) such that
\[
\|T_s \varphi - \varphi\|_\infty \leq \frac{\varepsilon}{\|R\|}, \quad s \in V_\varepsilon.
\]

Hence by (1) we obtain
\[
\|T_s R\varphi - R\varphi\|_\infty = \|R(T_s \varphi - \varphi)\|_\infty \leq \|R\| \|T_s \varphi - \varphi\|_\infty \leq \varepsilon,
\]
whence for a non-negative continuous function \( \eta_\varepsilon \) on \( G \) such that
\[\text{supp} \eta_\varepsilon \subset V_\varepsilon \quad \text{and} \quad \int_G \eta_\varepsilon \, d\mu = 1\]
we have
\[
\|R\varphi \ast \eta_\varepsilon - R\varphi\|_\infty \leq \varepsilon.
\]
Evidently, \( R\varphi \ast \eta \ast \in C_u(G) \), whence there exists a function \( \xi \in C_u(G) \) such that \( \xi(x) = R\varphi(x) \) for almost every \( x \). After modification (if necessary) on a set of the Haar measure zero we may assume that \( R\varphi \in C_u(G) \).

Next we show that \( C_0(G) \) is an invariant subspace of \( R \). The mapping \( \varphi \mapsto R\varphi(0) \) is a linear continuous functional on \( C_0(G) \). By the Riesz theorem, it is represented in the form

\[
R\varphi(0) = \int_G \varphi \, d\nu
\]

for some \( \nu \in M(G) \). Hence by (1) we obtain

\[
R\varphi(s) = (T_s R\varphi)(0) = (RT_s \varphi)(0) = \int_G \varphi(s + x) \, \nu(dx).
\]

Consequently, if \( \varphi \in C_c(G) \) and \( K \) is compact with

\[
|\nu|(G \setminus K) \leq \frac{\varepsilon}{\|\varphi\|},
\]

then for \( s \notin \text{supp} \varphi - K \) we have \( |R\varphi(s)| \leq \varepsilon \). Hence \( R\varphi \in C_0(G) \). For \( \varphi \in C_0(G) \) choose a sequence \( \varphi_n \in C_c(G) \) such that

\[
\lim_{n \to \infty} \|\varphi - \varphi_n\| = 0.
\]

Then

\[
\lim_{n \to \infty} \|R\varphi - R\varphi_n\| = 0 \quad \text{and} \quad R\varphi \in C_0(G),
\]

so \( C_0(G) \) is an invariant subspace of \( R \). Now for \( \varphi \in C_0(G) \) we can write

\[
\int_G \varphi(s) \, \nu(ds) = R\varphi(0) = R^2 \varphi(0) = \int_G R\varphi(s) \, \nu(ds) = \int_G \varphi(s + t) \, \nu(dt) \nu(ds),
\]

whence \( \nu \ast \nu = \nu \). Thus \( \hat{\nu}(\chi) = 1 \) or \( 0 \) for every \( \chi \in \hat{G} \), where \( \hat{\nu} \) denotes the Fourier transform of \( \nu \). Since the mapping \( \chi \mapsto \hat{\nu}(\chi) \) is continuous and \( \hat{G} \) is connected, we have \( \hat{\nu} \equiv 1 \) or \( \hat{\nu} \equiv 0 \) and \( R = E \) or \( R = 0 \), respectively, on \( C_0(G) \). In the case \( R = E \) on \( C_0(G) \) we have \( C_0(G) \subset \Phi \), whence \( \Phi = L^\infty(G) \). In the case \( R = 0 \) on \( C_0(G) \) we have \( \Phi \cap C_0(G) = \{0\} \). Thus the proof is completed.

Let \( M_0(G) \) denote the class of all closed subsets of \( G \) which contain a support of a non-zero measure belonging to \( M(G) \) with the Fourier transform vanishing at infinity. Every closed subset \( F \) of \( G \) such that \( \mu(F) > 0 \) belongs to \( M_0(G) \). In a non-discrete \( G \) there may exist \( F \in M_0(G) \) such that \( \mu(F') = 0 \) (cf. [3]).

**Theorem 2.** Let \( \Phi \) be a translation-invariant \(*\)-weakly closed proper subspace of \( L^\infty(G) \) and let \( \hat{G} \) be connected. If \( \sigma(\Phi) \in M_0(\hat{G}) \), then \( \Phi \) is uncomplemented in \( L^\infty(G) \).
Proof. Suppose that $\Phi$ is complemented in $L^\infty(G)$. By Theorem 1, $\Phi \cap C_0(G) = \{0\}$. Let $\nu$ be a non-zero measure supported by $\sigma(\Phi)$ such that $\hat{\nu} \in C_0(G)$. For every $f \in \text{An} \Phi$ we have $\langle f, \hat{\nu} \rangle = \langle \hat{f}, \nu \rangle = 0$, whence $\hat{\nu} \in \text{An} \text{An} \Phi = \Phi$. Thus $\hat{\nu}$ is a non-zero element of $\Phi \cap C_0(G)$. This contradiction proves our theorem.

**Theorem 3.** Let $I$ be a non-trivial ideal in $L^1(G)$ and let $\mathcal{G}$ be connected. If $\sigma(\text{An} I) \in M_0(\mathcal{G})$, then $I$ is uncomplemented in $L^1(G)$.

**Proof.** If $P$ were a continuous projection of $L^1(G)$ onto $I$, then $Q = E - P'$ would be a continuous projection of $L^\infty(G)$ onto $\text{An} I$, contrary to Theorem 2.

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**REFERENCES**


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