for kernels such as
\[ K(g, \varphi) = \frac{1}{2g^{2\beta}(a + \log g)} \sin \varphi \]
where the family of ellipses is now
\[ y_1 = g^2 \cos \varphi, \]
\[ y_2 = g^2(a + \log g) \sin \varphi, \]
\[ \beta \geq 1, \alpha > 0. \]
It is enough to take into account result (1.2) (Coifman–Guzmán)
The proof can be seen in [16] if \( \alpha = 1 \) and in [14] if \( \beta = 2 \).

References
[14] - Bound for integrals of Dirichlet type depending upon parameters, applications, to be published.

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periodic, in which case the function $x \mapsto \int_0^x f(u) du - f(0)x$ is almost periodic.

There arises a natural question whether $W^1$ almost periodicity in the above theorem may be replaced by some other types of almost periodicity. We shall present a negative result in this respect, that will concern so-called $E^2$ almost periodicity, a stronger property than being simultaneously $B'$ almost periodic for all $p \geq 1$. The result will be somewhat ineffective as it is often the case of results employing a probabilistic argument.

2. Prerequisites. We utilize various classes of almost periodic functions on $R$. Aside from the usual (uniformly) almost periodic functions on $R$, there appear:

(i) the Stepanov $S^p$ almost periodic functions on $R$ $(1 \leq p < +\infty)$, i.e., measuring functions on $R$ that are limits of sequences of trigonometric polynomials in one of the seminorms

$$\|f\|_{S^p} = \sup \left\{ (2T)^{-1} \int_{-T}^T |f(x+u)|^p du : x \in R \right\} \quad (T > 0),$$

no matter in which one;

(ii) the Weyl $W^p$ almost periodic functions on $R$ $(1 \leq p < +\infty)$, i.e., measuring functions on $R$ that are limits of sequences of trigonometric polynomials in the seminorm

$$\|f\|_{W^p} = \lim_{T \to +\infty} \|f\|_{S^p};$$

(iii) the Besicovitch $B^p$ almost periodic functions on $R$ $(1 \leq p < +\infty)$, i.e., those measurable functions on $R$ that are limits of sequences of trigonometric polynomials in the seminorm

$$\|f\|_{B^p} = \lim_{T \to +\infty} \left( (2T)^{-1} \int_{-T}^T |f(u)|^p du \right)^{1/p};$$

(iv) the $E^p$ almost periodic functions on $R$ $(1 \leq p < +\infty)$, i.e., those measurable functions on $R$ that are limits of sequences of trigonometric polynomials in the seminorm

$$\|f\|_{E^p} = \inf \left\{ c > 0 : \lim_{T \to +\infty} (2T)^{-1} \int_{-T}^T \exp \left( \frac{|f(u)|}{c} \right) du \leq 2 \right\}.$$

It is easily verified that a measurable function $f$ on $R$ is $E^p$ almost periodic $(1 \leq p < +\infty)$ if and only if there exists a sequence $(p_n)$ of trigonometric polynomials such that for every $a > 0$

$$\lim_{n \to +\infty} \lim_{T \to +\infty} (2T)^{-1} \int_{-T}^T \exp(a |f(u) - p_n(u)|^p) du = 1.$$
function \( x \rightarrow \exp(i f(x)) \). By the argument theorem of Bohr, the proof will be complete upon showing that \( g \) is almost periodic.

Since \( \|g\|_{\infty} = 1 \), \( g \) has at least one non-zero Fourier coefficient, say \( \hat{g}(\mu) \\ (\mu \in \mathbb{R}) \). Let \( \varepsilon > 0 \) be given. Applying the lemma to the function \( g_{\mu}(x) = g(x) \exp(-i \mu x) \ \ (x \in \mathbb{R}) \), we see that there exist positive numbers \( a_i \ (i = 1, \ldots, n) \) with \( \sum_{i=1}^{n} a_i = 1 \) and real numbers \( s_i \ (i = 1, \ldots, n) \) such that

\[
\left\| \sum_{i=1}^{n} a_i T_{s_i} g_{\mu} - \hat{g}(\mu) g \right\|_{\infty} < \varepsilon.
\]

Since the expression on the left side is equal to

\[
\left\| \sum_{i=1}^{n} a_i T_{s_i} g_{\mu} - \hat{g}(\mu) g \right\|_{\infty};
\]

and, in view of (co), each function \( g T_{s_i} g_{\mu} \ (i = 1, \ldots, n) \) is almost periodic, we infer that \( \hat{g}(\mu) g \) is the uniform limit of almost periodic functions. This in turn implies immediately that \( g \) is almost periodic.

The proof is complete.

4. A negative result. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Suppose there is given an ergodic flow on \( \Omega \), i.e., a one-parameter group \( \{S_t : t \in \mathbb{R}\} \) of measure-preserving transformations of \( \Omega \) onto itself, with the following properties:

(i) the map \( \mathbb{R} \times \Omega \ni (t, \omega) \rightarrow S_t(\omega) \in \Omega \) is measurable relative to \( (\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}) \), where \( \mathcal{B}(\mathbb{R}) \) denotes the Borel \( \sigma\)-algebra of \( \mathbb{R} \);

(ii) given a random variable \( f \) on \( \Omega, f \circ S_t = f \) a.s. (almost surely) for all \( t \in \mathbb{R} \) implies \( f \) is constant a.s.

Let \( \{a_i\} \) be a sequence in \( \mathbb{F}^{-1} \) of rationals independent positive numbers. Suppose the flow \( \{S_t\} \) has, for each \( n \in \mathbb{N} \), an eigenfunction \( \theta_n \) corresponding to the eigenfrequency \( a_n/2\pi \), such that

\[ \theta_n \circ S_t = \exp(i a_n t) \theta_n \]

for all \( t \in \mathbb{R} \). Suppose, moreover, that the eigenfunctions \( \theta_n \) form a family of independent random variables each one uniformly distributed on \( T \) (the unit circle).

That the above assumptions can be fulfilled is seen as follows. Take \( T^n \) for \( \Omega \) with the Borel \( \sigma\)-algebra of \( T^n \) as \( \mathcal{F} \), and the direct product measure obtained from Lebesgue measure on each copy of \( T \) as \( \mathbb{P} \). Define an ergodic flow on \( \Omega \) by putting

\[ S_t(\omega) = (\exp(i a_{n_1} t) \omega_1, \exp(i a_{n_2} t) \omega_2, \ldots) \]

for every \( \omega = (\omega_1, \omega_2, \ldots) \in \Omega \). Eventually realize an eigenfunction \( \theta_n (n \in \mathbb{N}) \) of \( \{S_t\} \) as the projection from \( \Omega \) onto the \( n \)th copy of \( T \).

Let

\[ F = \sum_{n=1}^{\infty} a_n^2 \text{Im} \theta_n. \]

Define a stochastic process \( \{F_t\} \) by putting

\[ F_t = F \circ S_t \]

for all \( t \in \mathbb{R} \). Clearly, each sample path of \( \{F_t\} \) is a real almost periodic function with mean value zero. Giving \( \mathbb{F} \), set

\[ X_t = \int_0^t F_s \ du. \]

In the sequel, when speaking about an almost periodic (resp. \( S^p \) almost periodic, \( 1 \leq p < +\infty \), etc.) stochastic process we shall mean that almost all trajectories of the process are almost periodic (resp. \( S^p \) almost periodic, \( 1 \leq p < +\infty \), etc.).

The main result of this section is

**Theorem 3.** The process \( \{\exp(i X_t)\} \) is \( E^2 \) almost periodic. Almost none of its sample paths is almost periodic.

**Proof.** For each \( n \in \mathbb{N} \), set

\[ Y^{(n)} = \sum_{k=1}^{n} a_k \Re \theta_{k_n}. \]

Let \( Y \) be the limit of \( (Y^{(n)}) \) a.s.; the existence of the limit follows from the three series theorem.

We claim that for any \( \alpha > 0 \)

\[ (1) \quad \lim_{n \to \infty} E \exp\{\alpha \exp(i Y) - \exp(i Y^{(n)})^2\} = 1. \]

To prove the claim note first that the sequence in (1) is minorized by one. Thus we need only appropriate estimates from above. Since \( \exp(ix) - 1 \leq |x| \) for \( x \in \mathbb{R} \), we may write

\[ (2) \quad E \exp\{\alpha \exp(i Y) - \exp(i Y^{(n)})^2\} = E \exp\{\alpha \exp(i (Y - Y^{(n)})^2)\} \leq E \exp\{\alpha (Y - Y^{(n)})^2\} \]

for every \( \alpha > 0 \) and every \( n \in \mathbb{N} \).

Denote by \( \{a_n\} \) a Bernoulli sequence, i.e., a sequence of independent identically distributed random variables each one taking the value plus and minus one with equal probability. \( \Re \theta_n \) being a sequence of symmetric real-valued random variables not exceeding one in absolute value, the Kahane contraction principle (cf. [8], Th. 2.4.9) neatly applies so as to give
Since the process \( \{X_t\} \) has continuous sample paths, \( \sup \{X_t: t \in R\} \) is a well-defined random variable on \( \Omega \). \( \Omega \) is non-negative because \( X_0 = 0 \). Clearly, \( Z \) is \( \mathcal{A} \)-adapted.

For each \( t \in R \) and each \( n \in N \), put
\[
X_t^{(n)} = \sum_{k=1}^{n} a_k [\Re\theta_k - \Re(\exp(i a_k \theta_k))].
\]
We see that for every \( t \in R \), \( X_t \) is the pointwise limit of \( \{X_t^{(n)}\} \).

Given \( n \in N \), let
\[
Z_n = \sup \{X_t^{(n)}: t \in R\}.
\]
Since \( \{a_k: k \in N\} \) is a rationally independent set, it follows from Kronecker's theorem that
\[
Z_n = \sum_{k=1}^{n} a_k (1 + \Re\theta_k).
\]
Of course, \( Z_n \) is \( \mathcal{A} \)-adapted. Using the three series theorem, we easily derive from (4) that
\[
\lim_{n \to \infty} Z_n = +\infty \text{ a.s.}
\]
We claim that given \( n \in N \), there exists an \( \mathcal{A} \)-adapted random variable \( \tau_n \) on \( \Omega \) such that
\[
X_t^{(n)} \geq Z_n - 1 \text{ a.s.}
\]
Indeed, since for each \( n \in N \), the process \( \{X_t^{(n)}\} \) is \( \mathcal{B}(R) \otimes \mathcal{A} \)-measurable, the set
\[
\{(t, \omega) \in R \times \Omega: X_t^{(n)}(\omega) \geq Z_n(\omega) - 1\}
\]
projecting along \( R \) onto \( \Omega \), is \( \mathcal{B}(R) \otimes \mathcal{A} \)-measurable. Now the claim follows upon applying the section theorem of Meyer (cf. [4], Th. 2.44).

Given any \( n, m \in N \), put
\[
\tau_n^m = \begin{cases} 
-m & \text{if } \tau_x \leq -m, \\
\tau_x & \text{if } -m < \tau_x < m, \\
m & \text{if } m \leq \tau_x. 
\end{cases}
\]
Since \( ||X_t^{(n+1)} - X_t^{(n)}||_{\infty} \leq \alpha_{n+1} + m \) for all \( n, m, \in N \), \( X_t^{(n)} \) is the \( L^\infty(\Omega) \) limit of \( \{X_t^{(n)}\} \), and so \( E^{\mathcal{A}}(X_t^{(n)}) \) is the \( L^\infty(\Omega) \) limit of \( (E^{\mathcal{A}}(X_t^{(p)}))_{p \geq n} \). But \( E^{\mathcal{A}}(X_t^{(p)}) \) = \( X_t^{(p)} \) a.s. for \( p \geq n \). Therefore
\[
E^{\mathcal{A}}(X_t^{(m)}) = X_t^{(m)} \text{ a.s.}
\]
On the other hand, we have
\[ Z \geq X_{\epsilon}^{(m)} \text{ a.s.} \]
Hence by (7)
\[ E^{\epsilon*}(Z) \geq X_{\epsilon}^{(m)} \text{ a.s.} \] (8)
Here we have applied the generalized conditional expectation operator to the non-negative possibly non-integrable \( Z \) (a very readable discussion of generalized conditional expectations including generalized martingale theorems may be found in [6], § 20). Letting \( m \) in (8) tend to infinity, we get
\[ E^{\epsilon*}(Z) \geq X_{\epsilon}^{(m)} \text{ a.s.} \]
Hence, in view of (5) and (6)
\[ Z = \lim_{\epsilon \to 0} E^{\epsilon*}(Z) = +\infty \text{ a.s.} \]
Since the sample paths of the process \( \{X_t\} \) are integrals of almost periodic functions having mean value zero, it easily follows from the latter formula and the argument theorem of Bohr that almost no trajectory of the process \( \{|X_t|\} \) is almost periodic.

The proof is complete.

We close the paper by remarking that the assumption made throughout that \( \Omega \) be a state space for an ergodic flow may be dispensed with. By a standard argument currently used in the theory of stationary processes, we may easily widen the scope of Theorem 3 so as to yield the following

**Theorem 4.** Let \( (\alpha_n) \) be a sequence in \( l^2 \) of rationally independent positive numbers. Given a probability space \( (\Omega, \sigma, \mathbb{P}) \), suppose \( \{\theta_n\} \) is a sequence of independent random variables on \( \Omega \) each one uniformly distributed on \( T \). Let
\[
F_t = \sum_{k=1}^n \alpha_k^2 \text{Im}(\exp(i\alpha_k t \theta_k)),
\]
\[
X_t = \int_0^t F_u \, du
\]
for all \( t \in \mathbb{R} \). Then each sample path of the process \( \{F_t\} \) is a real almost periodic function with mean value zero. Moreover, the process \( \{|X_t|\} \) is \( E^2 \) almost periodic while almost none of its sample paths is almost periodic.

The details of the proof of this theorem are left to the reader.

**References**