THE UNIVERSITY of ADELAIDE

School of Economics

Working Papers

ISSN 2203-6024

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Working Paper No. 2017-15
December 2017

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Abstract

This study examines an agency problem between a firm and its research employees in a continuous-time dynamic setting. As a solution to the problem, the study presents an optimal contract and discusses its implementation. In the implementation, a primary component of the agent's compensation is a risky security, and the principal lets the agent choose the consumption and effort levels subject to a sequence of minimum holding requirements. This implementation theoretically justifies the widespread use of performance-vested equity-based compensation by firms that rely on R&D.

Key words: Dynamic Contract, Repeated Moral Hazard, R&D, Performance-vesting Provisions, Private Saving

JEL: D23, D82, D86, J33, L22, O32

*I would like to thank Srihari Govindan, Ayca Kaya, Kyungmin Kim, Mandar Oak, B. Ravikumar, Raymond Riezman, Yuzhe Zhang, and seminar participants at The University of Iowa, the University of Adelaide, 2017 Asia Meeting of the Econometric Society in Hong Kong, and 2017 China Meeting of the Econometric Society in Wuhan for their valuable advice and comments. Any errors are my own.

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1 Introduction

Equity-based compensation for research employees has been a widely used strategy among new-economy firms. In fact, in cash-constrained start-up firms, equity-based compensations have become the primary component of compensation for employees in R&D departments. In 2014, Twitter, a successful start-up spent 26% of its revenue on equity-based compensations for its R&D employees. In terms of provision, equity-based compensations vest over time. Over the last two decades, equity-based rewards with performance-vesting provisions have become quite popular and are likely to surpass traditional simple time-vesting provisions in use. Between 1998 and 2012, performance-vested equity-based compensations to top executives in large U.S. companies grew from 20% to 70%. In R&D-intensive firms, the researchers' actions greatly influence firm performance, which in turn affects the return on equities. Thus, equity-based compensation directly links researchers' wealth to company performance. This link becomes stronger with performance-vesting, which ties vesting to the achievement of a series of performance objectives. The close connection between company performance and researchers' wealth realized by these compensation schemes reduces the agency problem faced by these firms, and incentivizes researchers to invest additional effort. Against this backdrop, the most pertinent question for these firms is whether this compensation scheme is optimal.

To answer this question, we study the contracting problem between a firm and its research employees in the abstract, derive the optimal contract, and present an implementation of the optimal contract. Finally, we compare the implementation results with the existing business practices. Findings indicate that the optimal contract can be implemented by using a risky security with a sequence of minimum holding requirements, which shares features with performance-vested equity-based compensations. These results thus provide a theoretical justification for the use of performance-vested equity-based compensation in firms that rely on R&D.

Briefly, the contracting problem is conceptualized as follows. A principal hires an agent to perform a multi-stage R&D project. The multi-stage feature captures the condition that the performance of research employees is usually linked to the completion of a sequence of milestones rather than their day-to-day practice. At any point in time, the agent can choose whether to put in effort or shirk. Subject to the agent's investing effort, the transition from one stage to the next is a Poisson-type process with a constant arrival rate. If the agent chooses to shirk, the Poisson arrival rate is zero. The principal cannot observe the agent's action. However, the progress of the innovation process is publicly observable, and the principal will use precisely this information to provide incentives optimally. To overcome the repeated moral-hazard problem, the principal offers the agent a long-term contract that specifies a flow of payments based on the principal's observation of the project outcome.

Recursive techniques are used to characterize the optimal dynamic contract. In the optimal contract, the principal uses a compensation scheme that combines punishments with rewards. If the agent fails, his payment decreases over time until he completes a stage. If the agent completes a stage, the principal rewards the agent by a discrete increase in the payment.

This paper also presents an implementation of the optimal contract, in which a primary component of the agent's compensation is a state-contingent security whose return is higher in case of success than in case of failure. At any point in time, besides the effort choice, the agent also chooses how much to consume and how much to invest in the security for the purpose of savings, subject to a minimum-holding requirement. Different from the optimal contract, in which the principal controls the agent's consumption directly, the agent chooses the consumption process in this implementation, which nonetheless generates the same effort and consumption process as the optimal contract.

This implementation is similar to the equity-based compensation scheme used in real-world settings. First, the return on the state-contingent security and equity price have a similar trend, with an notable increase after each breakthrough in R&D. Second, in this implementation, the agent is required to hold a certain amount of the state-contingent security until he completes the entire project. The required minimum holding amount reduces once the agent completes an innovation and progresses to the next stage. Similarly, equity-based grants used in practice usually have a vesting period during which they cannot be sold. Under the performance-vesting provision, a part of the equity-based grants is vested after the researchers achieve each predetermined performance target. By capturing these two main features, this implementation provides a theoretical rationale for the compensation schemes used in reality.

Finally, the robustness of the plan is evaluated by allowing the agent to save privately. In the optimal contract, the incentive for exerting more effort is partially derived from the deductions in the agent's compensation for unsatisfactory performance. With the risk of compensation cuts, the agent is likely to engage in private saving, if he can, due to a precautionary saving incentive.

However, if the return on private saving is very low, then the agent's precautionary saving incentive is diminished, which may restore the optimality of the contract that forbids private saving. We provide a sufficient condition under which the private saving problem can be ignored. For a special utility function, we are able to derive a closed-form upper-bound of the rate of return on private saving that satisfies the no-saving condition.

This paper is related to four strands of literature: management compensation, contracting for innovation, continuous-time dynamic contracting, and private saving in dynamic contracting. The management-compensation literature provides extensive research on equity-based grants. For instance, with regard to researchers' compensation, Anderson, Banker, and Ravindran (2000), Ittner, Lambert, and Larcker (2003), and Murphy (2003) have documented that executives and employees in new-economy firms receive more equity-based compensation than their counterparts in old-economy firms. Sesil, Kroumova, Blasi, and Kruse (2002) compared the performance of 229 new-economy firms offering broad-based stock options with that of their non-stock option counterparts. They showed that the former have higher shareholder returns. For performance-vesting provisions, Bettis, Bizjak, Coles, and Kalpathy (2010), using a sample of 983 equity-based awards that included either an accelerated- or a contingent-vesting provision tied to firm performance, found that performance-vesting provisions specify meaningful performance hurdles and provide significant incentives. Also, performance-vesting firms had significantly better subsequent operating performance than control firms. Bettis, Bizjak, Coles, and Kalpathy (2016) reported that performance-vesting provisions are displacing time-vesting provisions and can significantly amplify the incentive to take risk. Our model shows that firms use equity-based compensation to encourage researchers to bear some risks in return for incentives, and the minimum holding requirements in the implementation guarantee the minimum amount of risks that the researchers have to take for the purpose of incentives. When the R&D project progresses, the uncertainty of the project reduces and hence the minimum holding requirement is relaxed. This trend is consistent with the empirical finding that the performance-vesting provision provides better incentives by encouraging risk-taking actions. To the best of our knowledge, no theoretical work has derived a role specifically for performance-vesting provisions. This implementation bridges that gap in literature by providing a rationale of the growing use of performance-vesting provisions from a theoretical point of view.

A few researchers have investigated the topic of contracting for innovation. Manso (2011) studied a two-period model in which a principal provides incentives for an agent not only to work rather than shirk but also to work on exploration of an uncertain technology rather than exploitation of a known technology. Hörner and Samuelson (2013) and Bergemann and Hege (2005) studied contracting problems with dynamic moral hazard and private learning about the quality of the innovation project. Halac, Kartik, and Liu (2016) introduced adverse selection about the agent's ability into the problem. Shan (2017) studies a similar contracting problem when the R&D department consists of multiple agents. The current paper not only characterizes the optimal dynamic contract, but also provides an implementation in which the agent makes both effort and consumption decisions.

This paper also contributes to the rich and growing literature on continuous-time dynamic contracting. Sannikov (2008) analyzed a continuous-time principal-agent model, in which the output is a diffusion process with drift determined by the agent's unobserved effort. A similar Brownian motion framework is often used to model agency problems in fields such as corporate finance (DeMarzo and Sannikov (2006); He (2009); He (2011)). Recently, a few scholars have studied the dynamic moral hazard problem using a Poisson process, where the agent exerts unobservable effort that controls the arrival rate. In Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2015), bad events happen with higher Poisson arrival rate when agents do not put enough effort to prevent such events. In Sun and Tian (2017), the principal needs to provide incentive for the agent to exert effort to raise the arrival rate of a Poisson process. Most of these studies have assumed that the agent's risk is neutral. The risk-neutrality assumption implies that the agent does not receive any payment until the continuation utility reaches a payment threshold (Biais, Mariotti, Rochet, and Villeneuve (2010); Myerson (2015)), or only receives bonuses upon arrivals (Sun and Tian (2017)). In the present model, the agent is risk-averse. Besides providing incentive to work, the optimal contract also needs to account for consumption smoothing. Therefore, the agent's payment is contingent on the entire history and varies over time.

Private saving is one of the biggest challenges in dynamic contracting literature. Rogerson (1985) notes that if the agent is allowed access to credit, he would choose to save some of his wages, if he could, because of a wedge between the agent's Euler equation and the inverse Euler equation implied by the principal's problem. He (2012) characterizes the optimal contract when the agent who is assumed to have CARA utility is allowed to save. He shows that the principal cannot cut the agent's compensation for unsatisfactory performance if the agent can save privately at the same rate of return as the principal. Mitchell and Zhang (2010) study the unemployment insurance problem when the agent has hidden savings. In this paper, we provide a sufficient condition under which optimality of the contract obtained without private saving is not compromised even if the agent can save privately.

The rest of the paper is organized as follows. Section 2 describes the model. The optimal contract is characterized in Section 3. In Section 4, we present an implementation of the optimal dynamic contract. Section 5 discusses the private-saving problem and a case when the agent can succeed by "luck." Section 6 presents the conclusions.

2 The model

We consider a dynamic principal-agent model in continuous time. At time 0, a principal hires an agent to complete an R&D project. This project has N stages, which must be completed sequentially, i.e. to develop the stage n $(0 < n \leq N)$ innovation, the agent must have completed the innovations of stage $n-1$. The transition from one stage to the next is modeled by a Poisson-type process, which is affected by the agent's choice of effort. For simplicity, the agent is assumed to have only two effort choices: he can either put in effort or shirk. If the agent puts in effort, the probability that during a period of length $\triangle t$ the agent has not completed any innovation is $e^{-\lambda \triangle t}$, where λ is the Poisson arrival rate. If the agent chooses to shirk, he fails with probability 1, and the Poisson arrival rate is equal to zero.[1]

The agent's action cannot be monitored by the principal. However, the principal can observe exactly when each stage of the R&D project is completed. Let H^t be the history of the agent's performance up to time t. It records the number of stages completed and the time taken by the agent to complete each stage. By assumption, H^t is publicly observable, which is the only information that the principal can use to provide incentives to the agent.

At time 0, the principal offers the agent a contract that specifies a flow of consumption $c_t(H^t)$ based on the principal's observation of the agent's performance. Let T denote the stochastic stopping time when the agent completes the last-stage innovation. After time T, the principal does not need to provide any incentive for the agent to work, and hence the agent receives a constant payment over time, which is equivalent to a lump-sum consumption transfer at time T.

We assume that the agent's utility function has a separable form $U(c) - L(a)$, where $U(c)$ is the utility from consumption, and $L(a)$ is the disutility of exerting effort. We assume that

[1]We will relax this assumption and consider the case in which the agent can succeed by "luck" in Section 5.

$U : [0, +\infty) \to [0, +\infty)$ is an increasing, concave, and C^2 function. The agent's choice of effort is binary, indicated by $a \in \{0, 1\}$. $a = 1$ means that the agent chooses to put in effort, and $a = 0$ means that the agent chooses to shirk. Moreover, the disutility of putting in effort equals some $l > 0$, and the disutility of shirking equals zero, i.e., $L(1) = l$ and $L(0) = 0$.

Given the contract, at any time t, the agent makes the effort choice based on the observation of H^t. The effort process is denoted as $a = \{a_t(H^t), 0 \leq t < \infty\}$. The agent's objective is to choose the effort process a to maximize the total expected utility. Thus, the agent's problem is

$$\max_{\{a_t, 0 \leq t < +\infty\}} E\bigg[\int_0^T re^{-rt}(U(c_t) - L(a_t))dt + e^{-rT}U(c_T)\bigg],$$

where r is the discount rate.[2] Moreover, the agent has a reservation-utility v_0. If the maximum expected utility he can get from the contract is less than v_0, then the agent will reject the principal's offer.

For simplicity, we assume that the agent and the principal have the same discount rate. Hence, the principal's expected cost is given by

$$E\bigg[\int_0^T re^{-rt}c_t dt + e^{-rT}c_T\bigg].$$

We assume that the completion of R&D is quite valuable to the principal; therefore, he always wants to induce the agent to work. Hence, the principal's objective is to minimize the expected cost by choosing an incentive-compatible payment scheme subject to delivering the agent the requisite initial value of expected utility v_0. Therefore, the principal's problem is

$$\min_{\{c_t, 0 \leq t < +\infty\}} E\bigg[\int_0^T re^{-rt}c_t dt + e^{-rT}c_T\bigg]$$

s.t.

$$E\bigg[\int_0^T re^{-rt}(U(c_t) - l)dt + e^{-rT}U(c_T)\bigg] \geq v_0.$$

Finally, to simplify the analysis, we could recast the problem as one where the principal directly transfers utility to the agent instead of consumption. In the transformed problem, the principal chooses a stream of utility transfers $u_t(H^t)$ $(0 \leq t < +\infty)$ to minimize the expected cost of implementing positive effort. Then, the principal's problem becomes

[2] We normalize the flow term by multiplying it by the discount rate so that the total discounted utility equals the utility flow when the flow is constant over time. Thus, the agent's total discounted utility at time T equals $U(c^T)$.

$$\min_{\{u_t, 0 \le t < +\infty\}} E\left[\int_0^T re^{-rt} S(u_t)dt + e^{-rT} S(u_T)\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(u_t - l)dt + e^{-rT} u_T\right] \ge v_0,$$

where $S(u) = U^{-1}(u)$, which is the principal's cost of providing the agent with utility u. It can be shown that $S(u)$ is a decreasing and strictly convex function.

3 Optimal contract

To derive the optimal contract, we employ the standard approach described in the contracting literature: the optimal contract is written in terms of the agent's continuation utility v_t, which is the total utility that the principal expects the agent to derive at any time t. At any moment in time, given the continuation utility, the contract specifies the agent's utility flow, the continuation utility if he completes an innovation, and the law of motion of the continuation utility if he fails.

To derive the recursive formulation of this contracting problem, we first look at a discrete-time approximation of the continuous-time problem. The continuous-time model can be interpreted as the limit of discrete-time models in which each period lasts for Δt. When Δt is small, subject to effort exertion, the probability that the agent successfully completes an innovation during Δt is approximately $\lambda \Delta t$. The one-period discount factor is approximately equal to $\frac{1}{1+r\Delta t}$. For any given continuation utility v, the principal needs to decide on a triplet of factors $(u, \underline{v}, \bar{v})$ in each period, where

- u is the instantaneous payment in the current period;
- $\underline{v}$ is the next-period continuation utility if the agent fails to complete an innovation during this period;
- $\bar{v}$ is the next-period continuation utility if the agent completes an innovation during this period.

If the agent chooses to exert effort, his expected utility in the current period is

$$r(u - l)\Delta t + \frac{1}{1 + r\Delta t}[(1 - \Delta t \lambda)\underline{v} + \Delta t \lambda \bar{v}],$$

where the first term is the current-period utility flow, and the second term is the discounted expected continuation utility.

If the agent chooses to shirk, he does not incur any utility cost but will fail with probability 1. Thus, his expected utility in the current period becomes

$$ru\Delta t + \frac{1}{1+r\Delta t}\underline{v}.$$

The triplet $(u, \underline{v}, \bar{v})$ should satisfy two conditions. First, this contract should indeed guarantee that the agent gets the promised utility v. That is

$$r(u-l)\Delta t + \frac{1}{1+r\Delta t}[(1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}] = v.$$

Second, the contract should incentivize positive effort, i.e., the expected utility of putting in effort should be higher than the expected utility of shirking. Thus,

$$r(u-l)\Delta t + \frac{1}{1+r\Delta t}[(1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}] \geq ru\Delta t + (1-r\Delta t)\underline{v}.$$

Let $C_n(v)$ be the principal's minimum expected cost of providing the agent with continuation utility v at stage n. Then, $C_n(v)$ satisfies the following Bellman equation:

$$C_n(v) = \min_{u,\underline{v},\bar{v}} rS(u)\Delta t + \frac{1}{1+r\Delta t}[(1-\lambda\Delta t)C_n(\underline{v}) + \lambda\Delta t C_{n+1}(\bar{v})]$$

s.t.

$$\begin{aligned}
r(u-l)\Delta t + \frac{1}{1+r\Delta t}[(1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}] &= v,\\
r(u-l)\Delta t + \frac{1}{1+r\Delta t}[(1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}] &\geq ru\Delta t + \frac{1}{1+r\Delta t}\underline{v},
\end{aligned}$$

where $S(u)$ is the principal's cost of offering the instantaneous payment u.

To derive the Hamilton-Jacobi-Bellman (HJB) equation in continuous time, we multiply the Bellman equation by $(1+r\Delta t)$ and subtract $C_n(v)$ from each side to get

$$rC_n(v)\Delta t = \min_{u,\underline{v},\bar{v}} (1+r\Delta t)rS(u)\Delta t + [C_n(\underline{v}) - C_n(v)] + \lambda\Delta t[C_{n+1}(\bar{v}) - C_n(\underline{v})].$$

We divide the equation by Δt and let Δt converge to 0. Then, $\underline{v}$ converges to v, and the equation becomes

$$rC_n(v) = \min_{u,\bar{v}} rS(u) + C_n'(v)\frac{dv}{dt} + \lambda[C_{n+1}(\bar{v}) - C_n(v)],$$

where the left-hand side is the flow of the principal's costs, which is the sum of instantaneous payoff, the change in costs brought by the variation of continuation utility, and the change in costs when R&D enters the next stage at rate λ.

Performing a similar operation to the promise-keeping condition gives

$$\frac{dv}{dt} = rv - r(u-l) - \lambda(\bar{v} - v).$$

The promise-keeping condition becomes the evolution of the agent's continuation utility. In continuous time, therefore, the continuation utility changes smoothly in case of failure, and the rate of change is determined by u and $\bar{v}$. The continuation utility can be explained as the value that the principal owes the agent. Hence, it grows at the discount rate r and falls because of the flow of repayment $r(u-l)$ plus the gain of utility $\bar{v} - v$ at rate λ if the agent completes an innovation.

When Δt converges to 0, the incentive-compatibility constraint becomes

$$\lambda(\bar{v} - v) \geq rl.$$

To induce the agent to put in positive effort, the continuation utility should be raised by at least $\frac{rl}{\lambda}$ in case of success. The term $\frac{rl}{\lambda}$ is the minimum reward that the principal should give the agent when he completes the project. It is determined by three parameters: r, l, and λ, which have the following interpretations: (1) r is discount rate. The agent discounts the future utility at a higher rate when r is bigger. (2) l measures the cost of conducting research. When l is larger, the cost of conducting research is higher. (3) λ measures the complexity of the R&D project. A small λ implies a small chance of success. Thus, a big reward is associated with a high discount rate, a high cost of conducting research, or a low chance of success.

Thus, the principal's problem in continuous time is given by the following HJB equation:

$$rC_n(v) = \min_{u,\bar{v}} rS(u) + C_n'(v)\frac{dv}{dt} + \lambda[C_{n+1}(\bar{v}) - C_n(v)]$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ \lambda(\bar{v} - v) &\geq rl. \end{aligned}$$

As the agent is assumed to have limited liability, the continuation utility cannot be less than 0, because the agent can guarantee a utility level of 0 by not putting in any effort. Therefore, a negative continuation utility is not viable.

In the Appendix, we use a diagrammatic analysis to characterize the solution of the HJB equation. The main properties of the optimal contract are summarized in Proposition 3.1.

Proposition 3.1 *The principal's expected cost at any point is given by an increasing and convex function $C_n(v)$, which satisfies*

$$rC_n(v) = rS(u) + C_n'(v)[r(v-u)] + \lambda[C_{n+1}(\bar{v}) - C_n(v)],$$

and the boundary condition $C_n(0) = \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda}$. The instantaneous payment u satisfies $S'(u) = C_n'(v)$. When the agent completes an innovation, he enters the next stage and starts with the continuation utility $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda}$. In case of failure, the continuation utility v smoothly decreases over time and asymptotically reduces to 0. The utility flow u exhibits the same dynamics as the continuation utility v. The minimum-cost functions satisfy $C_n(v) > C_{n+1}(v)$ for all $v \geq 0$.

Proposition 3.1 indicates that the optimal contract combines rewards and punishments. The principal rewards the agent by an upward adjustment in the compensation after each success and punishes the agent by cutting his compensation for unsatisfactory performance. Figure 1 gives a sample path of the continuation utility for a 3-stage project. Thus, the principal induces the risk-averse agent to bear some risks by introducing some uncertainties into his compensation. Otherwise, the agent lacks an incentive to work. Proposition 3.1 also shows that the cost of delivering the same level of continuation utility is higher at an earlier stage of the project (Figure 2). This is because, at an earlier stage, the uncertainties about the future are higher. Hence, the cost of delivering the same level of continuation utility to a risk-averse agent is higher.

4 Implementation

The optimal contract presented in the previous section relies entirely on continuation utility, which is a highly abstract concept. Moreover, in the previous version, the principal controls the agent's consumption directly, i.e., the agent consumes all the payments from the principal at any point in time. In this section, we present an implementation of the optimal contract, in which a primary component of the agent's compensation is a state-contingent security. In this implementation, besides choosing between exerting effort and shirking, the agent also makes consumption

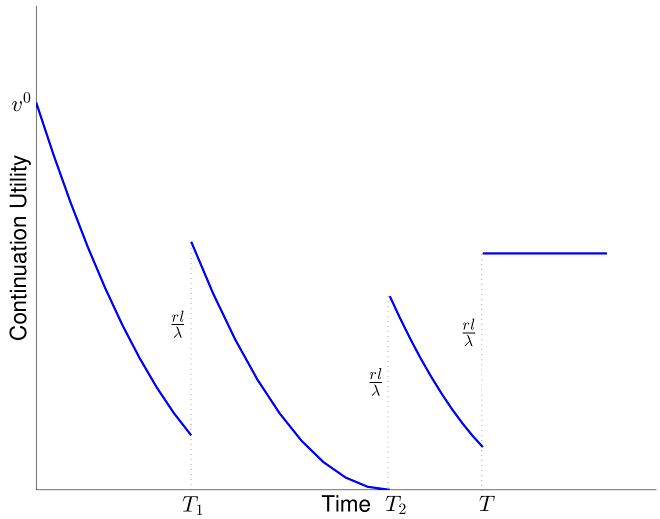

Figure 1: A sample path of continuation utility.

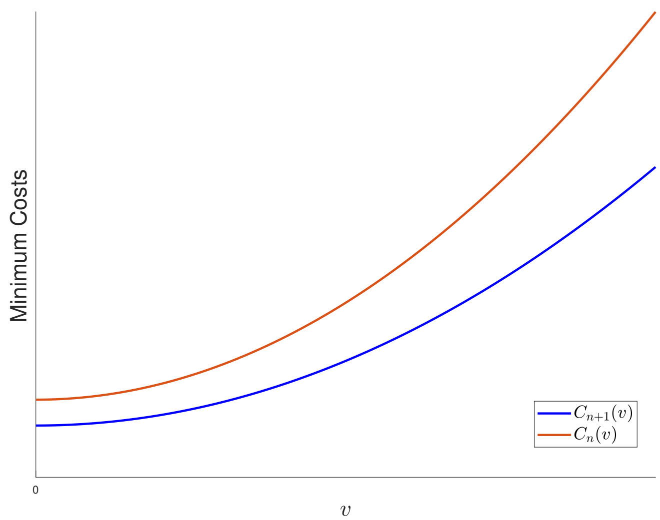

Figure 2: Cost functions.

decisions. Yet, the implementation generates the same consumption allocation as the original optimal contract. Finally, we discuss how this implementation relates to the compensation schemes used in reality.

To implement the contract, before the project starts, the principal provides the agent with initial wealth y^0, and a part of it is paid in the form of a state-contingent security. The return on the security when the project succeeds is higher than the return when the project fails. At any point in time, the agent can invest in this security for saving purpose subject to a minimum holding requirement $\underline{y}_n$. Note that the minimum holding requirement depends on the stage level n. To describe how the value of the state-contingent security evolves over time, we first look at a discrete-time approximation of the continuous-time setting. Suppose in period t the project is at stage n. The agent decides to hold a certain amount of securities whose value in the next period $t+1$ equals y_{t+1} if the project fails during that period. The value of this amount of securities in case of success in period $t+1$ is given by $Y_{n+1}(y_{t+1})$, where $Y_{n+1}(y_{t+1})$ is a function of y_{t+1}. The value of these securities in the current period t is determined by fair-price rule, i.e., the value equals the expected present value, which is given by $P_n(y_{t+1}) = \frac{1}{1+r\Delta t}[(1-\lambda\Delta t)y_{t+1} + \lambda\Delta t Y_{n+1}(y_{t+1})]$. In other words, if the agent allocates $P_n(y_{t+1})$ of his current wealth to the security, in next period, his wealth level equals y_{t+1} in case of failure and $Y_{n+1}(y_{t+1})$ in case of success. For easier tracking of the agent's wealth, we write the value of the agent's security in the current period and in the next period in case of success as functions of its value in the next period in case of failure. Now, besides effort choice, the agent also chooses how much to consume and how much to invest in the security to carry his wealth to the next period. Let y_t denote the agent's wealth in period t. Then, his budget constraint is

$$rc_t\Delta t + P_n(y_{t+1}) \leq y_t,$$

where the first term on the left-hand side is his consumption in the current period, and the second term is his investment in the security if he wants a guaranteed wealth level of y_{t+1} in case of failure in the next period. If Δt converses to 0, we can derive the evolution of the agent's wealth in case of failure, which satisfies

$$\frac{dy}{dt} = ry - rc - \lambda[Y_{n+1}(y) - y].$$

Thus, when the project is at stage n, the agent's wealth in case of failure grows at rate r and decreases because of consumption spending c and the loss on investment in security $\lambda(Y_{n+1}(y) - y)$. If the agent succeeds, his wealth grows to $Y_{n+1}(y)$.

Now, the agent's problem is to choose an effort process and a consumption plan that maximize his discounted expected utility. Let $V_n(y)$ be the maximum expected utility that the agent can get given wealth level y when the project is at stage n. In recursive form, the agent's problem is to solve the following HJB equation

$$rV_n(y) =$$
$$\max\{\max_c r[U(c) - l] + V_n'(y)\frac{dy}{dt} + \lambda[V_{n+1}(Y_{n+1}(y)) - V_n(y)], \max_c rU(c) + V_n'(y)\frac{dy}{dt}\}$$

s.t.

$$\begin{aligned} \frac{dy}{dt} &= ry - rc - \lambda[Y_{n+1}(y) - y], \\ y &\geq \underline{y}_n. \end{aligned}$$

The next proposition shows that this implementation generates the same consumption allocation and effort choice as the original optimal contract when the principal sets the initial wealth, payoff in case of success, and minimum holding requirement appropriately. The proof is in the appendix.

Proposition 4.1 *Suppose the principal provides the agent with initial wealth y^0*

$$y^0 = C_1(v^0),$$

and at stage n

$$Y_{n+1}(y) = C_{n+1}\left(C_n^{-1}(y) + \frac{rl}{\lambda}\right),$$
$$\underline{y}_n = C_n(0).$$

Then, given income y, the highest discounted expected utility the agent can get is

$$V_n(y) = C_n^{-1}(y),$$

and he chooses the same consumption process as the one in the optimal contract and always exerts effort until he completes the last-stage innovation. The minimum holding requirement satisfies $\underline{y}_n > \underline{y}_{n+1}$.

The premise of this implementation lies in the fact that the agent's utility maximization problem is the dual problem of the principal's cost minimization problem in Section 3. Given continuation utility v, $C_n(v)$ is the minimum expected cost to finance the incentive-compatible compensation scheme. From the dual perspective, given the expected wealth $y = C_n(v)$, the maximum expected utility that the agent can reach should equal v. Further, the consumption allocation should be the same. In this implementation, the agent invests in the risky security for saving purpose, and hence the return on savings is state contingent. When the state-dependent rates of return are chosen appropriately, the agent's Euler equation mimics the inverse Euler equation implied by the principal's problem. In other words, the wedge between the Euler equation and the inverse Euler equation, as stated in Rogerson (1985), disappears.

In this implementation, the state-contingent security plays a key role in incentives. As discussed in Section 3, the principal has to let the agent bear some risks; otherwise the agent will shirk his work. In the implementation, the risks are embedded in the state-contingent security. The gap between the value in case of success and in case of failure guarantees that the agent will exert effort. The minimum holding requirement is the lowest level of risk that can incentivize the agent to exert effort. Proposition 4.1 shows that the minimum holding requirement is relaxed after each innovation. This is because when the project progresses to the next stage, the uncertainty of the project reduces, and the minimum level of risks to be borne by the agent for incentive purposes also becomes less.

In the financial market, there is no asset that has the exact same payoff structure as the state-contingent security used in this implementation. However, the equity of a company is a reasonable proxy. Since these firms rely heavily on R&D, the performance of the employees in the R&D units greatly influences the firms' performance outcomes, which closely links employees' performance and the return on firms' equities. In particular, each breakthrough in R&D is followed by a notable increase in the firm's equity price. The absence of such developments in a firm over a period generally leads to a drop in its equity price. Thus, among all available assets, the company's equity has the closest payoff-pattern to the state-contingent security. Another feature of this implementation is the sequence of minimum holding requirements that the agent has to meet until the completion of the project. The minimum holding requirement is relaxed after each success. In the real-world, this feature is mimicked by using performance-vesting provisions, under which a part of equity grants is vested when the research employee achieves a predetermined performance target.

In the last two decades, equity-based grants have become the most popular compensation scheme among new-economy firms (see Anderson, Banker, and Ravindran (2000); Ittner, Lambert, and Larcker (2003); Murphy (2003)). Within equity-based compensation, performance-vesting provisions are replacing traditional time-vesting provisions. Performance-vesting provisions directly link vesting to the achievements of performance targets, which significantly amplifies the researchers' incentive to take risk (Bettis, Bizjak, Coles, and Kalpathy (2016)). The similarities between the compensation scheme used in practice and our implementation of the optimal contract suggest that firms are as close to optimality as is allowed by the market structure. In other words, this implementation provides a theoretically sound rationale for the widespread use of performance-vested equity-based compensation in firms that rely on R&D.

5 Extensions

This section examines two extensions of the model. First, we study the case in which the agent can save privately. We provide a sufficient condition under which allowing private saving does not affect the optimality of the contract. Next, we analyze a case when the agent can succeed by "luck."

5.1 Private saving

In previous sections, we assumed that the agent cannot save privately. In Section 3, the contract determines a consumption path contingent on the agent's performance. At any point in time, the agent consumes all the payments from the principal and cannot save or borrow. In Section 4, although the agent chooses how much to consume, he can only invest in the state-contingent security for saving purpose. An important feature of the optimal contract in Section 3 is that the principal punishes the agent by cutting his consumption in case of unsatisfactory performance. A well-known result, first documented by Rogerson (1985), shows that the optimal contract is impracticable if the agent can save privately due to a precautionary saving incentive.[3] Aware of the risk of lower compensation in case of failure, a risk-averse agent would save some of his income for consumption smoothing purpose. In some cases, the agent may adopt a double-deviation strategy by shirking to avoid the costs of working and saving privately to smooth consumption, which makes the problem even more complicated for the principal.

To see how private saving affects the optimal contract derived in the previous sections, we first examine the case when the agent can save privately at the same rate of return r as the principal.

[3]This problem only arises when the agent can save privately. If the principal can monitor the agent's saving, then the principal can offer a contract contingent on the agent's saving.

For illustration, consider the following one-period deviation in the discrete-time approximation. Suppose from time t to $t+\Delta t$, instead of working and consuming all the payments received from the principal, the agent shirks and saves some of the payments at time t and consumes the saving at $t+\Delta t$. Because of shirking, the agent will fail. The marginal effect of shifting consumption in this way is

$$-ru'(c_t)\Delta t + \frac{1}{1+r\Delta t}[ru'(\underline{c}_{t+\Delta t})\Delta t](1+r\Delta t) = r\Delta t[u'(\underline{c}_{t+\Delta t}) - u'(c_t)] > 0.$$

The inequality is due to the result that the principal cuts the agent's compensation in case of failure so that $\underline{c}_{t+\Delta t} < c_t$ and the assumption that the agent is risk averse. This result suggests that if the agent shirks then he could receive higher utility through private saving. Under the consumption allocation of the optimal contract, the agent is indifferent to working or shirking because the incentive-compatibility condition is always binding. It further implies that if the agent shirks and shifts some consumption from the current period to the next period, his deviation payoff is higher than the payoff on the equilibrium path. Therefore, if the agent can save privately at the same rate as the principal, the principal cannot punish the agent by cutting his compensation for unsatisfactory performance. Otherwise, the agent will adopt a double-deviation strategy, and the optimal contract becomes invalid. This result is similar to the observation in He (2012).

However, if the agent incurs a cost on account of hiding his saving, then the low return on private saving will mitigate the agent's precautionary saving incentive. If the return is considerable low, it may restore the optimality of the contract derived in the previous sections. Note that the agent's saving incentive depends on his marginal utility of consumption. To simplify the notation, we use m_t, where $m_t = U'(c_t)$, to denote the agent's marginal utility of consumption at any time t given the contract. Suppose the agent can save privately at rate r'. The following proposition provides sufficient condition under which the agent has no incentive to save.

Proposition 5.1 *Given contract $\{c_t(H^t), 0 < t < +\infty\}$, if in case of failure the agent's marginal utility of consumption satisfies*

$$\frac{d\ln m_t}{dt} \leq -(r'-r),$$

then the agent has no incentive to save privately. At any point in time, he consumes all the payments from the principal and exerts efforts until the project is completed.

Proposition 5.1 indicates that if the return on private saving is very low so that $r' \leq r - \frac{d\ln m_t}{dt}$ for all $\{c_t(H^t), 0 < t < +\infty\}$, then the contract derived in Section 3 is still optimal in as the agent will not deviate from the consumption path suggested by the principal and always put effort at work. In a general setting, this sufficient condition is difficult to ascertain because it has to be held at any time t on all possible consumption paths. However, note that

$$\frac{d\ln m_t}{dt} = \frac{d\ln U'(c_t)}{dt} = \frac{U''(c_t)}{U'(c_t)}\frac{dc_t}{dt}.$$

If the agent utility function has CARA form, then $\frac{U''(c_t)}{U'(c_t)}$ is a constant number. It can be shown that $\frac{dc_t}{dt}$ is bounded.[4] Therefore, for CARA utility function, there exists an upper bound of r' such that the sufficient condition in Proposition 5.1 is satisfied. Next, we show that when the agent's utility function is logarithmic, we are able to derive a closed-form solution of the contracting problem. It enable us to find a closed-form upper bound of r', which satisfies the sufficient condition in Proposition 5.1.

Example with Logarithmic Utility

Consider an example in which the agent's utility from consumption is $U(c) = \ln c$, and the project has infinitely many stages. Because the project has infinitely many stages, the cost function C does not depend on stage level and satisfies the following HJB equation[5]

$$rC(v) = \min_{u,\bar{v}} rS(u) + C'(v)\frac{dv}{dt} + \lambda[C(\bar{v}) - C(v)]$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ \lambda(\bar{v} - v) &\geq rl. \end{aligned}$$

In the Appendix, we solve this problem and obtain a closed-form solution. The property of the contract is summarized in the following proposition.

[4]Note that $c_t = S(u_t)$ and u_t is determined by $S'(u_t) = C'(v_t)$. Hence, c_t is a continuous function of the continuation utility v_t. Proposition 3.1 shows that $\frac{dv_t}{dt} = r(v_t - u_t)$. Therefore, $\frac{dv_t}{dt}$ is also a continuous function of v_t. The highest level of continuation utility that the agent can achieve is $v_0 + \frac{Nrl}{\lambda}$ when the agent completes all N innovations instantly. Therefore, $\frac{dv_t}{dt}$ is bounded. This implies that $\frac{dc_t}{dt}$ is bounded because c_t is a continuous function of v_t.

[5]We consider infinitely many stages because it is easier to derive the closed-form solution when the cost function C does not depend on stage level.

Proposition 5.2 *When the agent's utility from consumption is $U(c) = \ln c$ and the project has infinitely many stages, the minimum cost of delivering continuation utility v is given by $C(v) = e^{v+\frac{\lambda}{r}(e^{\frac{rl}{\lambda}}-1)}$. When the agent completes an innovation, he enters the next stage and starts with continuation utility $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda}$. In case of failure, the continuation utility v evolves according to $\frac{dv}{dt} = -\lambda(e^{\frac{rl}{\lambda}} - 1)$. The contract remains optimal even if the agent can save privately as long as the return on private saving is not higher than $r - \lambda(e^{\frac{rl}{\lambda}} - 1)$.*

When the agent has logarithmic utility function, the implementation takes a very simple form. Given wealth y, the agent's expected utility is given by

$$V(y) = C^{-1}(y) = \ln y - \frac{\lambda}{r}(e^{\frac{rl}{\lambda}} - 1).$$

According to Proposition 4.1, in case of success, the agent's wealth increases to

$$Y(y) = C\left(C^{-1}(y) + \frac{rl}{\lambda}\right) = e^{\frac{rl}{\lambda}}y,$$

which is a linear function of y. In case of failure, the agent's wealth evolves according to

$$\frac{dy}{dt} = ry - rc - \lambda[Y(y) - y] = [r - \lambda(e^{\frac{rl}{\lambda}} - 1)]y - rc.$$

Hence, the risky security that implements the contract has a very simple structure: 1) in cases of success, the value of the security rises by $e^{\frac{rl}{\lambda}}$ times; 2) in case of failure, the rate of return equals $r - \lambda(e^{\frac{rl}{\lambda}} - 1)$. To implement the optimal contract, the principal only needs to offer the agent a wealth level of $e^{v_0+\frac{\lambda}{r}(e^{\frac{rl}{\lambda}}-1)}$ before the project starts, and then let the agent decide his consumption and effort choice. Proposition 4.1 shows that the agent will choose the exact same consumption path as the one in the optimal contract. It is obvious that if the return on private saving is not higher than return on the security in case of failure, $r - \lambda(e^{\frac{rl}{\lambda}} - 1)$, then the agent will not have any incentive to engage in private saving and deviate from the optimal consumption path. The interpretation of this result is that when the firm adopts equity-based compensation and the return on the equities is higher than the return on private saving, then the employees prefer to hold the equities for saving instead of saving privately. Thus, the firms can almost mimic the optimal contract even if they cannot monitor their employees' saving levels.

5.2 Success by "luck"

So far, we have assumed that the agent fails with a probability of 1 if he shirks. This assumption implies that a success unambiguously informs the principal that efforts have been exerted. In this subsection, we relax this assumption and consider the case in which the agent can succeed even without exerting effort. Let λ be the arrival rate of success when the agent exerts effort, and λ' ($0 < \lambda' < \lambda$) be the arrival rate when the agent shirks. By exerting effort, the agent increases his chance of success from λ' to λ. Accordingly, his benefit for exerting effort is $(\lambda - \lambda')(\bar{v} - v)$, and his costs of exerting effort is rl. The incentive-compatibility condition is given by

$$(\lambda - \lambda')(\bar{v} - v) \geq rl.$$

The HJB equation of the principal's problem becomes

$$rC_n(v) = \min_{u,\bar{v}} rS(u) + C_n'(v)\frac{dv}{dt} + \lambda[C_{n+1}(\bar{v}) - C_n(v)]$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ (\lambda - \lambda')(\bar{v} - v) &\geq rl. \end{aligned}$$

By doing a similar diagrammatic analysis, we can characterize the solution to the HJB equation. The properties of the optimal contract are summarized in the following proposition.

Proposition 5.3 *The principal's expected cost at any point is given by an increasing and convex function $C_n(v)$ that satisfies the HJB equation and the boundary condition*

$$C_n\left(\frac{\lambda' l}{\lambda - \lambda'}\right) = \frac{\lambda C_{n+1}\left(\frac{(r+\lambda')l}{\lambda - \lambda'}\right)}{r + \lambda}.$$

When the agent completes an innovation, the agent's continuation utility increases to $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda - \lambda'}$. In case of failure, the continuation utility v decreases over time and asymptotically goes to $\frac{\lambda' l}{\lambda - \lambda'}$. The instantaneous payment u has the same dynamics as the continuation utility v.

The only difference between this case and the case when the agent fails for sure if he shirks is the positive lower bound on the implementable continuation utility $\frac{\lambda' l}{\lambda - \lambda'}$. To provide an incentive, the principal should reward the agent by raising his continuation utility by $\frac{rl}{\lambda - \lambda'}$ after success. Even if the agent shirks, he still can receive the reward by "luck" and thus guarantee a positive expected utility $\underline{v}$ which satisfies:

$$\underline{v} = \int_{t=0}^{\infty} e^{-rt} e^{-\lambda' t} \lambda' \left(\underline{v} + \frac{rl}{\lambda - \lambda'}\right) dt.$$

The expression on right-hand side of the equation is the agent's expected utility when he shirks continuously and receives 0 instantaneous payment until a success arrives by "luck". In this case, the utility flow is always 0, and the continuation utility increases from to $\underline{v}$ to $\underline{v} + \frac{rl}{\lambda - \lambda'}$ at rate λ'. Solving the equation, we obtain the lower bound of continuation utility $\underline{v} = \frac{\lambda' l}{\lambda - \lambda'}$. Because an agent can guarantee this level of utility, the principal cannot punish the agent too severely. Otherwise, the agent will choose to shirk and wait to get lucky.

6 Conclusion

To examine the optimality of the equity-based compensation scheme that is widely used by new-economy firms for their research employees, we study a contracting problem in which a principal hires an agent to perform a multi-stage R&D project. The R&D process is modeled by a Poisson process. In the optimal contract, incentive is offered to the agent in two ways: (1) the principal raises the agent's compensation to a higher level when he successfully completes an innovation (reward); (2) if the agent fails to complete the innovation, his compensation decreases continuously over time (punishment). We show that the optimal contract could be implemented using a risky security, whose return depends on the outcome of the project. The agent is required to meet a sequence of minimum holding requirements until he completes the project. The minimum holding requirements are relaxed when the project progresses to the next stage. In this implementation, instead of the principal directly controlling the agent's consumption as in the optimal contract, the agent chooses both consumption level and effort level. By a duality argument, we show that this implementation yields the same consumption allocation as the one in the optimal contract. By incorporating the main features of the performance-vested equity-based compensation scheme, this implementation provides a theoretical justification for the widespread use of the scheme to compensate research employees. We also consider the case in which the agent can save privately. If the agent incurs a cost when he saves privately so that the return on private saving is very low, then the agent has no incentive to engage in private saving, and the optimality of the contract and implementation remain intact.

Appendix

Proof of Proposition 3.1

Since the contracting problem is similar to the single agent problem studied in Shan (2017), we use a similar proof to prove this proposition.

We first show that C_n satisfies the following statement for all n ($0 < n \leq N+1$) by an induction argument.

Statement A: *C_n is a C^1 function. Its derivative, C_n', is a continuous and strictly increasing function and satisfies that $C_n'(v) \geq S'(v)$ for all $v > 0$, and $C_n'(0) = S'(0)$.*

Step 1: C_{N+1} satisfies Statement A

When the agent completes the last stage innovation, he receives a lump-sum transfer. Hence, $C_{N+1} = S$, which satisfies Statement A.

Step 2: Derive the phase diagram in the v-$C_n'(v)$ plane assuming that C_{n+1} satisfies Statement A

Suppose that C_{n+1} satisfies Statement A. The HJB equation of the stage-n problem is

$$rC_n(v) = \min_{u,\bar{v}} rS(u) + C_n'(v)\frac{dv}{dt} + \lambda[C_{n+1}(\bar{v}) - C_n(v)]$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ \lambda(\bar{v} - v) &\geq rl. \end{aligned}$$

To characterize the solution of the HJB equation, we do a diagrammatic analysis in the v-$C_n'(v)$ plane. Given a point $(v, C_n'(v))$ in the plane, $(u, \bar{v})$ are determined by the following Kuhn-Tucker conditions:

$$S'(u) - C_n'(v) + \mu = 0, \quad (1)$$

$$\lambda C_{n+1}'(\bar{v}) - \lambda C_n'(v) + \gamma\lambda = 0, \quad (2)$$

$$\lambda(\bar{v} - v) \geq rl, \quad (3)$$

$$u \geq 0, \quad (4)$$

$$\gamma[\lambda(\bar{v} - v) - rl] = 0, \tag{5}$$

$$\mu u = 0. \tag{6}$$

where equation (1) and (2) are first-order conditions, (3) is the incentive-compatibility condition, and γ and μ are Lagrangian multipliers which satisfy $\gamma, \mu \leq 0$. Using the envelop theorem, we can derive that

$$\frac{dC'_n(v)}{dt} = \gamma\lambda,$$

which determines the dynamics of $C'_n(v)$ at a given point. It implies that the dynamics of $C'_n(v)$ depend on whether the incentive-compatibility constraint is binding or not. The dynamics of v are given by

$$\frac{dv}{dt} = rv - r(u - l) - \lambda(\bar{v} - v).$$

The next two lemmas provide the dynamics at any point in the v-$C'_n(v)$ plane.

Lemma A.1 *In the phase diagram, the dynamics of $C'_n(v)$ satisfy:*

$$\frac{dC'_n(v)}{dt} \begin{cases} = 0, & \text{if } C'_n(v) \geq C'_{n+1}(v + \frac{rl}{\lambda}); \\ < 0, & \text{if } C'_n(v) < C'_{n+1}(v + \frac{rl}{\lambda}). \end{cases}$$

Proof of Lemma A.1: If $C'_n(v) \geq C'_{n+1}(v + \frac{rl}{\lambda})$, the incentive constraint is slack. This is because if we let $\gamma = 0$, then the first-order condition (2) becomes $C'_{n+1}(\bar{v}) = C'_n(v)$. Then, $C'_n(v) \geq C'_{n+1}(v + \frac{rl}{\lambda})$ implies that $\bar{v} \geq v + \frac{rl}{\lambda}$, and hence the incentive-compatibility constraint is satisfied. This verifies that the solution is $\gamma = 0$ and $C'_{n+1}(\bar{v}) = C'_n(v)$, and the incentive constraint is slack. In this case, the dynamics of $C'_n(v)$ satisfy $\frac{dC'_n(v)}{dt} = \gamma\lambda = 0$. In the other case, if $C'_n(v) < C'_{n+1}(v + \frac{rl}{\lambda})$, then incentive-compatibility constraint must be binding, and $\gamma < 0$. Otherwise, the first-order-condition (2) implies that $C'_{n+1}(\bar{v}) = C'_n(v) < C'_{n+1}(v + \frac{rl}{\lambda})$, and hence $\bar{v} < v + \frac{rl}{\lambda}$, which violates the incentive-compatibility constraint. In this case, $\frac{dC'_n(v)}{dt} = \gamma\lambda = \lambda[C'_n(v) - C'_{n+1}(v + \frac{rl}{\lambda})] < 0$. *Q.E.D.*

Lemma A.2 *In the phase diagram, the dynamics of v satisfies:*

$$\frac{dv}{dt} \begin{cases} < 0, & \text{if } C'_n(v) > S'(v); \\ = 0, & \text{if } C'_n(v) = S'(v); \\ > 0, & \text{if } C'_n(v) < S'(v). \end{cases}$$

Proof of Lemma A.2: If $C'_n(v) > S'(v)$, then the first-order condition (1) implies that $S'(u) = C'_n(v)$, and hence $u > v$. Then,

$$\frac{dv}{dt} = rv - r(u-l) - \lambda(\bar{v} - v) \leq rv - ru < 0,$$

where the first inequality is because of the incentive-compatibility condition.

Next, suppose $C'_n(v) = S'(v)$. As C_{n+1} satisfies Statement A, we have $C'_n(v) = S'(v) \leq C'_{n+1}(v) < C'_{n+1}(v + \frac{rl}{\lambda})$. Thus, the incentive-compatibility constraint is binding by Lemma A.1. The first-order condition (1) implies that $S'(u) = C'_n(v)$, and hence $u = v$. Thus,

$$\frac{dv}{dt} = rv - r(u-l) - \lambda(\bar{v} - v) = rv - ru = 0.$$

Similarly, if $C'_n(v) < S'(v)$, then the incentive-compatibility constraint binds, and (1) implies that $u < v$. Then

$$\frac{dv}{dt} = rv - r(u-l) - \lambda(\bar{v} - v) = rv - ru > 0.$$

Q.E.D.

These two lemmas show that the $C'_n(v) = C'_{n+1}(v + \frac{rl}{\lambda})$ locus determines the dynamics of $C'_n(v)$: $C'_n(v)$ is constant over time above it and decreasing over time below it. The $C'_n(v) = S'(v)$ locus determines the dynamics of v: v is decreasing over time above it and increasing over time below it. The dynamics are summarized in Figure 3.

Step 3: Derive the optimal path

In this step, we search for the optimal path in the phase diagram. By Cauchy-Lipschitz theorem, there is an unique path from any $v_0 > 0$ to the point $(0, S'(0))$ (Path 1 in Figure 4). Any path on which the state variable v diverges to infinity could be ruled out (such as Path 2). This rules out any paths in the area below Path 1. In the area above Path 1, the continuation utility v is decreasing over time. When v hits the lower bound 0, it cannot decrease any further. Thus, we must have $dv/dt \geq 0$ at $v = 0$. This condition rules out any paths above Path 1 (such as Path 3) because on such paths $dv/dt < 0$ when v reaches 0. Then, Path 1 is the only candidate path left in the phase diagram, and it is the optimal path that we are looking for. The final step is to pin down the boundary condition at $v = 0$. At this point, $u = 0$ and $\bar{v} = \frac{rl}{\lambda}$. Thus, when v reaches 0, the agent's continuation utility and instantaneous payment remain at 0 until he completes an innovation. To incentivize the agent to put in positive effort, the principal needs to increase the

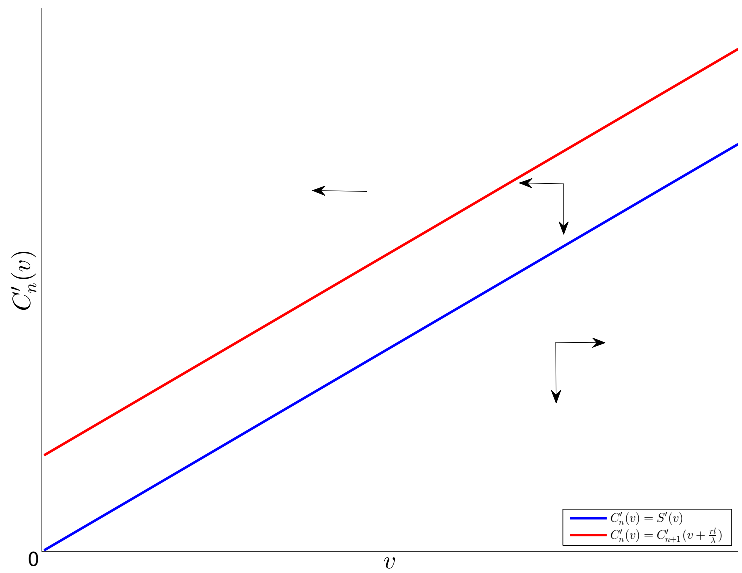

Figure 3: Phase Diagram

agent's continuation utility to $\frac{rl}{\lambda}$ once the agent completes the current stage innovation. Therefore, the boundary condition at $v = 0$ satisfies

$$C_n(0) = \int_{t=0}^{\infty} e^{-rt} e^{-\lambda t} \lambda C_{n+1}\left(\frac{rl}{\lambda}\right) dt = \frac{\lambda C_{n+1}\left(\frac{rl}{\lambda}\right)}{r + \lambda}.$$

The optimal path and the boundary condition together determine the solution of the HJB equation.

Step 4: C_n also satisfies Statement A

From Step 3, the optimal path locates between the $C_n'(v) = S'(v)$ locus and the $C_n'(v) = C_{n+1}'(v + \frac{rl}{\lambda})$ locus, and it reaches the lower bound of the continuation utility at the $(0, S'(0))$ (path 1 in Figure 4). Therefore, $C_n'(v) \geq S'(v)$ for all $v > 0$, and $C_n'(0) = S'(0)$. Moreover, $C_n'(v)$ is a continuous increasing function. Therefore, C_n satisfies Statement A. This step completes the induction argument, and hence C_n satisfies Statement A for all n $(0 < n \leq N + 1)$.

On the optimal path, $C_n'(v)$ is strictly increasing in v, which implies that $C_n(v)$ is strictly convex. In addition, $C_n'(v) \geq S'(v) > 0$ for all $v \geq 0$, which implies that $C_n(v)$ is an increasing function. Note that the optimal path locates in the area where the incentive-compatibility constraint binds.

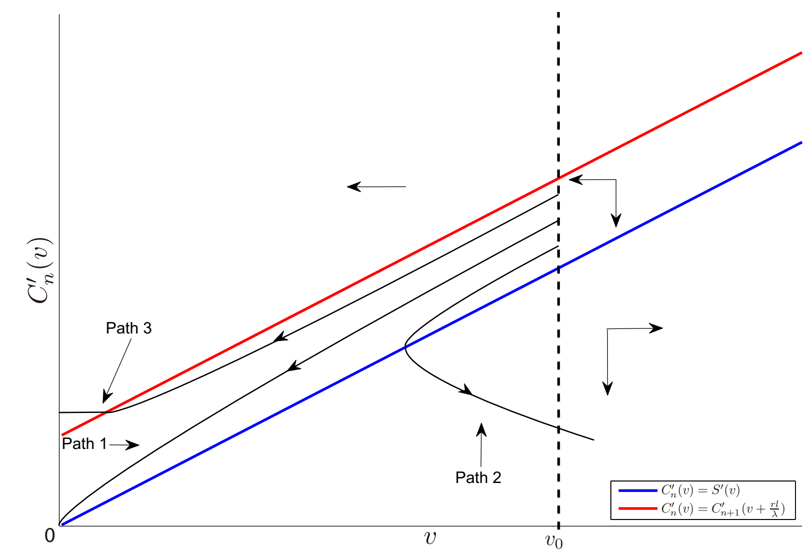

Figure 4: Optimal Path

Hence, $\bar{v} = v + \frac{rl}{\lambda}$. On the optimal path, v is decreasing over time and asymptotically converges to 0. The instantaneous payment flow is determined by the first-order condition $S'(u) = C'_n(v)$. As $S(u)$ and $C_n(v)$ are both convex, u and v are positively related. Thus, u has the same dynamics as v.

To show that minimum-cost functions satisfy $C_n(v) > C_{n+1}(v)$ for all $v \geq 0$, we prove the following statement by backward induction.

Statement B: *The minimum-cost functions satisfy: (1) $C_n(v) > C_{n+1}(v)$ for all $v \geq 0$; (2) $C'_n(v) > C'_{n+1}(v)$ for all $v > 0$ and $C'_n(0) = C'_{n+1}(0)$.*

At stage N, by Statement A, we have $C'_N(v) > C'_{N+1}(v) = S'(v)$ for all $v > 0$ and $C'_N(0) = C'_{N+1}(0) = S'(0)$. It also implies that $C_N(v) > C_{N+1}(v)$ for all $v \geq 0$ because $C_N(0) = \frac{\lambda S(\frac{rl}{\lambda})}{r+\lambda} > 0 = S(0) = C_{N+1}(0)$. Hence, the Statement B is verified for $n = N$.

Suppose Statement B is true for some stage n. On the optimal path, $dC'_n(v)/dt = \lambda[C'_n(v) - C'_{n+1}(v + \frac{rl}{\lambda})]$, and $dv/dt = r(v - u_n)$, where u_n satisfies $S'(u_n) = C'_n(v)$. Hence, in the phase diagram, the slope of C'_n at v satisfies

$$\frac{dC'_n(v)}{dv} = \frac{\lambda[C'_n(v) - C'_{n+1}(v + \frac{rl}{\lambda})]}{r(v - u_n)}.$$

Similarly, for C'_{n-1}, we have

$$\frac{dC'_{n-1}(v)}{dv} = \frac{\lambda[C'_{n-1}(v) - C'_n(v + \frac{rl}{\lambda})]}{r(v - u_{n-1})},$$

where u_{n-1} satisfies $S'(u_{n-1}) = C'_{n-1}(v)$. Suppose $C'_{n-1}(v) = C'_n(v)$ at some v. Then $S'(u_n) = S'(u_{n-1})$, and hence $u_n = u_{n-1}$. Furthermore, because $C'_n(v + \frac{rl}{\lambda}) > C'_{n+1}(v + \frac{rl}{\lambda})$ from the assumption that Statement B is true for stage n, it follows that

$$\frac{dC'_{n-1}(v)}{dv} > \frac{dC'_n(v)}{dv}.$$

Thus, if C'_{n-1} and C'_n intersect, C'_{n-1} cuts C'_n from below. This result implies that C'_{n-1} intersects C'_n at most once. We have shown that $C'_{n-1}(0) = C'_n(0) = S'(0)$. Therefore, $C'_{n-1}(v) > C'_n(v)$ for all $v > 0$. A direct implication of this result is that $C_{n-1}(v) > C_n(v)$ for all v because $C_{n-1}(0) = \frac{\lambda C_n(\frac{rl}{\lambda})}{r+\lambda} > \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda} = C_n(0)$. Hence, Statement B is also true for stage $n-1$. Then, by backward induction, the Statement B is true for all n $(0 < n \leq N)$.

Proof of Proposition 4.1

We first verify that $V_n(y) = C_n^{-1}(y)$ all n $(0 < n \leq N)$ is the solution to the HJB equation of the agent's problem under the conditions of Proposition 4.1. Since C_n is a strictly increasing and differentiable function, C_n^{-1} exists and is also differentiable. Suppose $V_n(y) = C_n^{-1}(y)$ and $V_{n+1}(y) = C_{n+1}^{-1}(y)$, then

$$\begin{aligned} \lambda[V_{n+1}(Y_{n+1}(y)) - V_n(y)] - rl &= \lambda[V_{n+1}(C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})) - V_n(y)] - rl \\ &= \lambda[C_n^{-1}(y) + \frac{rl}{\lambda} - C_n^{-1}(y)] - rl \\ &= 0. \end{aligned}$$

This result implies that if $V_n(y) = C_n^{-1}(y)$ is value function of the HJB equation of the agent's problem, then the agent is always indifferent between exerting effort and shirking. Thus, we have

$$RHS = rU(c^*) + V'_n(y)\frac{dy}{dt}$$

$$
\begin{aligned}
&= rU(c^*) + V_n'(y)\{ry - rc^* - \lambda[C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda}) - y]\} \\
&= rU(c^*) + \frac{(r+\lambda)y - rc^* - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C_n'(C_n^{-1}(y))},
\end{aligned}
$$

where c^* is the optimal choice of consumption which is determined by the first-order condition $U'(c^*) = \frac{1}{C_n'(C_n^{-1}(y))}$.

The next step is to find the expression for $\frac{1}{C_n'(C_n^{-1}(y))}$. Since, from principal's problem, $C_n(v)$ satisfies the following differential equation

$$
(r+\lambda)C_n(v) = rS(u^*) + C_n'(v)[r(v-u^*)] + \lambda C_{n+1}(v + \frac{rl}{\lambda}),
$$

then we have

$$
\frac{1}{C_n'(v)} = \frac{r(v-u^*)}{(r+\lambda)C_n(v) - rS(u^*) - \lambda C_{n+1}(v + \frac{rl}{\lambda})},
$$

where u^* is the optimal choice of utility flow that satisfies $S'(u^*) = C_n'(v)$. Taking $v = C_n^{-1}(y)$ into the equation above, we get

$$
\begin{aligned}
\frac{1}{C_n'(C_n^{-1}(y))} &= \frac{r[C_n^{-1}(y) - u^*]}{(r+\lambda)C_n(C_n^{-1}(y)) - rS(u^*) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})} \\
&= \frac{r[C_n^{-1}(y) - u^*]}{(r+\lambda)y - rS(u^*) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})},
\end{aligned}
$$

where $S'(u^*) = C_n'(C_n^{-1}(y))$. Since $S(u^*) = U^{-1}(u^*)$, it follows that $\frac{1}{U'(S(u^*))} = C_n'(C_n^{-1}(y))$. Because c^* satisfies $U'(c^*) = \frac{1}{C_n'(C_n^{-1}(y))}$, we have $U'(S(u^*)) = U'(c^*)$, and hence $S(u^*) = c^*$ and $u^* = U(c^*)$. Therefore,

$$
\frac{1}{C_n'(C_n^{-1}(y))} = \frac{r[C_n^{-1}(y) - U(c^*)]}{(r+\lambda)y - rc^* - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}.
$$

Taking this expression for $\frac{1}{C_n'(C_n^{-1}(y))}$ into the right-hand side of the HJB equation, we have

$$
\begin{aligned}
RHS &= rU(c^*) + \frac{(r+\lambda)y - rc^* - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C_n'(C_n^{-1}(y))} \\
&= rU(c^*) + r[C_n^{-1}(y) - U(c^*)] \\
&= rC_n^{-1}(y) \\
&= rV_n(y) \\
&= LHS.
\end{aligned}
$$

Thus, we have verified that $V_n(y) = C_n^{-1}(y)$ is the solution to the HJB equation of the agent's problem.

Next, we show that this implementation generates the same consumption allocation as the optimal contract. We have shown that the maximum expected utility that the agent can derive from the implementation for a given wealth level y is $V_n(y)$, and he is always willing to exert effort. $V_n(y)$ could also be interpreted as his "continuation utility" given wealth y. From the agent's HJB equation, his "continuation utility" evolves according to $\frac{dV_n(y)}{dt} = V_n'(y)\frac{dy}{dt} = rV_n(y) - rU(c^*)$, where c^* is the optimal choice of consumption given y. In the principal's problem, given continuation-utility v, the continuation utility evolves according to $\frac{dv}{dt} = rv - ru^*$, where u^* is the optimal choice of utility flow. From the previous proof, if $v = V_n(y)$ then $u^* = U(c^*)$. Therefore, given the same continuation utility, the implementation and the optimal contract choose the same consumption, which further induce the same law of motion of continuation utility. Finally, the initial condition that $y^0 = C_1(v^0)$ guarantees that the agent starts with initial continuation-utility v^0, and the minimum holding requirement $y \geq \underline{y}_n = C_n(0)$ guarantees that positive effort is always implementable. Thus, the implementation and the optimal contract generate the same consumption allocation under all possible realization of the agent's performance.

Finally, since $\underline{y}_n = C_n(0)$, and $C_n(0) > C_{n+1}(0)$ by Proposition 3.1, the minimum holding requirement satisfies $\underline{y}_n > \underline{y}_{n+1}$.

Proof of Proposition 5.1

To show that under the condition in Proposition 5.1 the agent will not engage in private saving, we check the agent's precautionary saving incentive at any time t. Since the contract punishes the agent by cutting his consumption in case of unsatisfactory performance, the lowest consumption path from time t to t' for the agent is the one when he fails to complete any innovation during this period time. Since the agent's utility function is concave, he has the strongest incentive to save when he receives this "worst" consumption path. Thus, if we can show that the agent has no incentive to save even on this "worst" consumption path, then it implies that the agent has no incentive to save on any other consumption paths. The marginal cost of saving at time t equals m_t. Since the rate of return on private saving is r', the marginal benefit of saving at time t and consuming it at t' is $e^{-r(t'-t)}e^{r'(t'-t)}m_{t'} = e^{(r'-r)(t'-t)}m_{t'}$. If in case of failure the agent's marginal utility of consumption satisfies

$$\frac{d \ln m_t}{dt} \leq -(r' - r),$$

then on this "worst" consumption path

$$\ln m_{t'} - \ln m_t \leq -(r' - r)(t' - t).$$

It implies that

$$\ln m_t \geq \ln m_{t'} + (r' - r)(t' - t).$$

Taking exponential to both sides, it becomes

$$m_t \geq e^{(r'-r)(t'-t)} m_{t'}.$$

Thus, the marginal cost of saving exceeds the marginal benefit. Therefore, the agent has no incentive to save on the "worst" consumption path. It further implies that the agent has no incentive to saving on any other consumption paths. Therefore, if $\frac{d \ln m_t}{dt} \leq -(r' - r)$ in case of failure, the private saving problem can be ignored. If the agent will not deviate from the consumption path offered by the principal, the incentive compatibility condition then guarantees that the agent will always exert effort.

Proof of Proposition 5.2

For logarithmic utility function $U(c) = \ln(c)$, the cost of delivering u is $S(u) = e^u$. We first guess that the cost function takes the form of qe^v $(q > 0)$—a constant times e^v. Then, we use it to solve the minimization problem on the right-hand side of the HJB equation. If the right-hand side also takes the form of a constant times e^v, then we can pin down the constant q from the HJB equation and the guess is verified.

Taking $C(v) = qe^v$ into the right-hand side of the HJB equation, we have

$$RHS = \min_{u,\bar{v}} re^u + qe^v \frac{dv}{dt} + \lambda q(e^{\bar{v}} - e^v)$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u - l) - \lambda(\bar{v} - v), \\ \lambda(\bar{v} - v) &\geq rl. \end{aligned}$$

Utility-flow u satisfies the first-order condition $S'(u) = C'(v)$. Therefore,

$$e^u = qe^v,$$

which implies $u = v + \ln q$.

The incentive compatibility constraint must be binding, otherwise first-order conditions imply that $\bar{v} = v$, which violates the incentive compatibility constraint. Hence, $\bar{v} = v + \frac{rl}{\lambda}$, which implies that $\frac{dv}{dt} = -r \ln q$. Taking the solution for u and $\bar{v}$ into the right-hand side of the HJB equation, it becomes

$$\begin{aligned} RHS &= rqe^v + qe^v(-r\ln q) + \lambda(qe^{v+\frac{rl}{\lambda}} - qe^v) \\ &= (rq - rq\ln q + \lambda qe^{\frac{rl}{\lambda}} - \lambda q)e^v. \end{aligned}$$

which also takes the form of a constant times e^v. Finally, letting the left-hand side of the HJB equation equal the right-hand side, we have

$$rq = rq - rq\ln q + \lambda qe^{\frac{rl}{\lambda}} - \lambda q.$$

Solving the equation, we get

$$q = e^{\frac{\lambda}{r}(e^{\frac{rl}{\lambda}}-1)}.$$

A nice property of logarithmic utility function is that the marginal utility from consumption satisfies

$$m = U'(c) = \frac{1}{c} = \frac{1}{e^u} = e^{-u}.$$

Hence, $\frac{d\ln m}{dt} = -\frac{du}{dt}$, and the no saving condition in Proposition 5.1 becomes that $\frac{du}{dt} \geq r' - r$. Since $u = v + \ln q$, we have

$$\frac{du}{dt} = \frac{dv}{dt} = -r\ln q = -\lambda(e^{\frac{rl}{\lambda}} - 1).$$

Thus, the no-saving condition becomes $-\lambda(e^{\frac{rl}{\lambda}} - 1) \geq r' - r$, which implies $r' \leq r - \lambda(e^{\frac{rl}{\lambda}} - 1)$. This result means that as long as the rate on private saving is not higher than $r - \lambda(e^{\frac{rl}{\lambda}} - 1)$, the agent has no incentive to save, and hence he will not deviate from the consumption path offered by the principal.

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