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# An Indirect Evolutionary Justification of Risk-Neutral Bidding in Fair-Division Games

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# An indirect evolutionary justification of risk neutral bidding in fair division games*

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**Abstract**

We justify risk neutral equilibrium bidding in commonly known fair division games with incomplete information in an evolutionary setup by postulating (i) minimal common knowledge assumptions, (ii) optimally responding agents to conjectural beliefs about how others behave and (iii) evolution of conjectural beliefs with fitness measured by expected payoffs. We axiomatically justify the game forms, derive the evolutionary games for first- and second-price fair division and determine the evolutionarily stable conjectures. The latter coincide with equilibrium bidding, irrespectively of the number of bidders, i.e., heuristic belief adaptation implies the same bidding behavior as equilibrium analysis based on common knowledge and counterfactual bids.

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# 1 Introduction

Unlike auctions, fair division considers closed group interaction: there is no outside party, like sellers (in standard auctions) or buyers (in procurement auctions) confronting bidders. Typical examples of such situations are divorce and inheritance settlements in private life, regulated in family law, and terminating joint venture firms with rules in commercial law. Güth and Van Damme (1986) and Cramton, Gibbons and Klemperer (1987) have modelled these as games under incomplete information and shown how they relate to bidding. McAfee (1992) and Morgan (2004) compare first- and second-price fair division games to other procedures and de Frutos (2000) extends the analysis to asymmetric cases. Moldovanu (2002) reviews the literature on partnership dissolution when private information is not independent. A common feature of these studies is that equilibrium bidding hinges upon a commonly known prior about bidders' private information. Following Vickrey (1961) and Harsanyi (1967/8), private value information is captured by a commonly known fictitious chance move about whose results the interacting parties are privately informed. This simply means that chance randomly selects a vector of individual values according to a commonly known probability distribution and that each party only learns about its own value. Altogether, private information is transformed in uncertainty about chance events and equilibrium bidding requires to bid not only for one's true value but also for others expected own values.

Our alternative modelling approach avoids such common knowledge assumption by embedding protagonists in an evolutionary framework. Bidders only know the number of competitors and their own private value but are unaware how private values are generated. Bidders cope with this realistic ambiguity of information by predicting the highest bids of their competitors as non-negative shares of their private values, i.e., by anchoring conjectural beliefs on their own private value, and by reacting optimally to their conjectural beliefs. Given such behavior one can determine for each possible value vector and constellation of individual conjectures what each party earns. We subject belief conjectures to evolutionary selection, measuring fitness by expected profits for an infinite population and randomly formed groups of bidders. For first- and second-price fair division bidding, the evolutionarily stable conjectural beliefs justify risk neutral equilibrium bidding, as derived in the literature.

Compared to Güth and Pezanis-Christou (2015), who derive a similar result for standard first-price auctions, fair division bidding is based on closed bidder groups. The presence of external agents renders mechanism design for auctions less restricted than for fair division. So second-price auctions are incentive compatible and feature truthful bidding as evolutionarily stable in case of private values, whereas second-price fair division is only underbidding proof.[1] Thus, our companion study could focus on the first-price auction format.

Section 2 justifies first- and second-price fair division game forms. Section 3 describes the resulting fair division games when assuming commonly known priors. Section 4 defines the evolutionary models and determines their evolutionarily stable conjectures. Section 5 concludes.

# 2 Game forms for fair division problems

Institutions typically define implementable mechanisms without guaranteeing information closedness as required by game-theoretic equilibrium analysis (Harsanyi, 1967/8 grants information closedness, i.e., com-

[1] See Weibull (1995) for the definition and analysis of evolutionarily stable strategies (ESS) which are essentially refined symmetric equilibria of symmetric games.

monly known games, by assuming a fictitious chance move). Actually the legal rules (in family and commercial laws) regulate fair division problems only via game forms, not by well-defined fair division games. In this section we first derive the game forms for fair division and then describe how they are complemented by a commonly known, possibly fictitious, chance move rendering them suitable for equilibrium analysis.

Objective and interpersonally verifiable fairness of allocation results can generally be guaranteed by relying only on *stated* values which are typically privately known, similar to voting rules letting election outcomes only depend on voting results. We refer to stated values as (monetary) bids and guarantee procedurally fair rules of bidding. One requirement is that all net exchanges are envy-free according to bids. To define this formally, consider fair division problems involving $n \geq 2$ bidders who collectively own, with equal property rights, an indivisible asset, e.g., a joint venture firm. Bidders evaluate the exclusive ownership of this asset by their private values $v_i \geq 0$ for $i = 1, .., n$, which are private information.

To bring about procedurally fair allocation results when having to terminate collective ownership by allowing one of the partners to attain exclusive ownership, each bidder $i = 1, .., n$ submits a bid $b_i \geq 0$. Using the vector notation $b = (b_1, .., b_n)$ for determining who of the $n$ bidders becomes the exclusive owner $w = w(b)$ and which compensation $t_i(b)$ should be paid by $w(b)$ to all non-winning bidders $i \neq w(b)$, envy-free net exchanges according to bids require:

- $t_i(b) = t(b)$ for all $i \neq w(b)$ and
- $b_{w(b)} - (n-1)t(b) \geq t(b) \geq b_i - (n-1)t(b)$ for all $i \neq w(b)$ and $i = 1, .., n$.

The first condition asks for equal compensation of all non-winners $i \neq w(b)$. The first weak inequality of the second condition asks that the winner $w(b)$ prefers his net exchange, according to his own bid $b_{w(b)}$ over the common compensation payment, $t(b)$. The second weak inequality requires non-winners $i \neq w(b)$ to prefer the compensation $t(b)$ over the $b_i$-evaluated net exchange of the winner $w(b)$. By adding $(n-1)t(b)$ and denoting $p(b) = nt(b)$, one obtains

$$b_{w(b)} \geq p(b) \geq b_i \qquad \text{for all } i \neq w(b)$$

Thus, for all possible bid vectors $b$,

i) winners $w(b)$ have to bid highest,

ii) prices $p(b)$ cannot exceed highest bids and cannot fall below second-highest bids, and

iii) prices are shared equally among all bidders.[2]

In line with much of the auction literature, we focus on the first- and second-price rule which both are not incentive compatible: the second-price rule is only underbidding-proof while the first-price one is only overbidding-proof. As for auctions, one can characterize the first-price rule by requiring not only envy-free but also equal net exchanges according to bids what implies $b_{w(b)} - (n-1)t(b) = t(b)$ or $b_{w(b)} = p(b)$ (see Güth, 2017).

[2] Van Damme (1985) argued that envy-freeness according to bids does not guarantee envy-free net exchanges according to values. But as values are only privately known this could not be checked interpersonally and objectively as required for legally implementable rules.

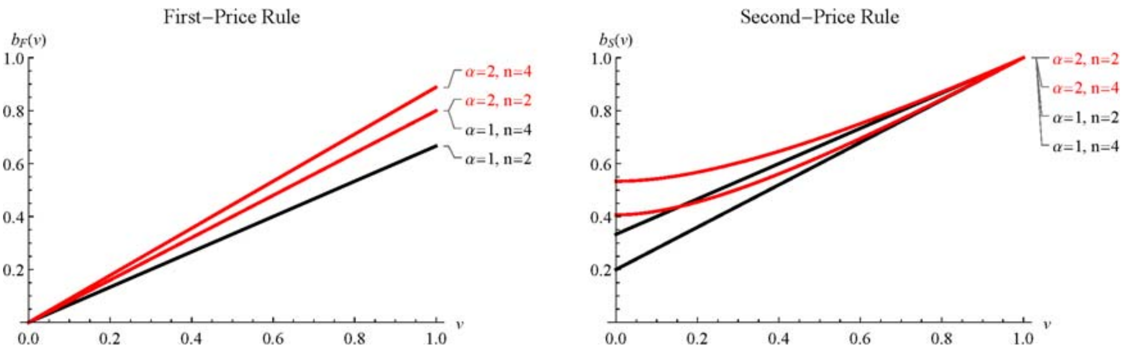


Figure 1: Nash equilibrium bidding strategies for first- and second-price fair division games.

To guarantee well-defined fair division games for both pricing rules, we rely on independent and identically distributed random values $v_i$ drawn (with replacement) from a commonly known distribution $F$ with density $f$ defined on $[\underline{v}, \overline{v}]$ and assume $\underline{v} = 0$ and $\overline{v} = 1$ for notational convenience. Due to this commonly known prior both pricing rules yield well-defined, informationally closed games in the tradition of Harsanyi (1967/8), whose equilibrium bidding strategies for the first- and second-price rule are (see Güth and Van Damme, 1986, and Cramton et al., 1987):

$$b_F(v_i) = \frac{\int_0^{v_i} x \,\mathrm{d}F(x)^n}{F(v_i)^n} \qquad \text{and} \qquad b_S(v_i) = v_i + \frac{\int_{v_i}^1 [F(x) - 1]^n \,\mathrm{d}x}{[F(v_i) - 1]^n}$$

Figure 1 visualizes the bidding strategies for the first- and second-price rule when $F$ is a Beta distribution $B(v, \alpha, 1)$ with density $f(v) = v^{\alpha-1}\Gamma(\alpha+1)/[\Gamma(\alpha)\Gamma(1)]$ defined on $(0, 1)$, and equal to the uniform distribution when $\alpha = 1$. When the second-price rule prevails, $b_S(v_i)$ is affine linear for $\alpha = 1$ and affine nonlinear otherwise whereas $b_F(v_i)$ is always linear. Further, the equilibrium strategies for the first-price rule may be identical for particular constellations of parameters characterizing $F$ and the number of bidders $n$.[3]

# 3 Indirect evolutionary analysis

As Güth and Pezanis-Christou (2015) for standard first-price sealed bid auctions with independent private values, we employ an indirect evolutionary approach. Bidders react optimally to simple beliefs predicting the highest bid of their competitors to be anchored on their own value $v_i$ via some positive coefficient. More specifically, assume that Nature faces an infinite population of bidders with randomly selected groups of $n$ bidders with each bidder $i$ bidding some fraction $q_i > 0$ of the value $v_i$, for $i = \{1, ..., n\}$. Nature keeps track of the realised average payoff of each $q_i$-type in each of many such encounters with $n - 1$ co-bidders and selects among conjectures (i.e., the various $q_i$-fractions) according to the realised average profits. The

[3]For example, when $F$ is a Beta distribution $B(v, \alpha, 1)$, the equilibrium bidding strategies for $\alpha = 1$ and $n = 4$ are identical to those for $\alpha = 2$ and $n = 2$, a pairwise equivalence which holds for $n = \{6, 8, 10, ...\}$ with $\alpha = 1$ and $n = \{3, 4, 5, ...\}$ with $\alpha = 2$.

question to answer is which bidding strategy will survive in such an evolutionary set-up. To answer this question, consider a $q$-monomorphic population with $q > 0$ (for all bidders) and a mutant with $p > 0$ which might invade the population. If this is impossible, then $q$ is evolutionarily stable.

### 3.1 First-price rule

The expected payoff of such $p$-mutant confronting a $q$-monomorphic population under the first-price rule is:

$$\Pi_F(p) = \int_{pv>qx} \left(v - \frac{n-1}{n} pv\right) \, \mathrm{d}F(x)^{n-1} + \int_{pv<qx} \frac{qx}{n} \, \mathrm{d}F(x)^{n-1}$$

or equivalently

$$\Pi_F(p) = \int_0^{\frac{pv}{q}} \left(1 - \frac{n-1}{n} p\right) v\mathrm{d}F(x)^{n-1} + \int_{\frac{pv}{q}}^{\infty} \frac{qx}{n} \, \mathrm{d}F(x)^{n-1}.$$

The first integral stands for $p$-mutant's expected profit from winning with a bid $pv$ whereas the second represents the payment it would receive from losing the contest. Taking the first-order condition with respect to $p$ yields:

$$\frac{n-1}{n} \int_0^{\frac{pv}{q}} \mathrm{d}F(x)^{n-1} - \frac{n-1}{n} v^2 F\left(\frac{pv}{q}\right)^{n-2} f\left(\frac{pv}{q}\right) + \frac{n - p(n-1)}{nq} (n-1) v^2 F\left(\frac{pv}{q}\right)^{n-2} f\left(\frac{pv}{q}\right) = 0.$$

For this mutant to invade the population, this first-order condition must satisfy $p = q$, i.e.,

$$\frac{n-1}{n} \int_0^{v} \mathrm{d}F(x)^{n-1} - \frac{n-1}{n} v^2 F\left(v\right)^{n-2} f\left(v\right) + \frac{n - q(n-1)}{nq} (n-1) v^2 F\left(v\right)^{n-2} f\left(v\right) = 0,$$

which is satisfied for

$$q_F^* = \frac{(n-1) v^2 F\left(v\right)^{n} f\left(v\right)}{(n-1) v^2 F\left(v\right)^{n} f\left(v\right) - \frac{(1-n) v F(v)^{n+1}}{n}}.$$

The two approaches, the equilibrium and the evolutionary one, yield the same predictions if $q_F^* = b_F(v_i)/v_i$. This condition holds in particular for the family of Beta distributions $B(v, \alpha, \beta)$ with $\beta = 1$. In this case, $q_F^* = n\alpha/(1 + n\alpha)$, which is indeed equal to $b_F(v_i)/v_i$.

### 3.2 Second-price rule

As the equilibrium bidding strategy for the second-price rule implies overbidding on $(0, 1)$ but not at $\overline{v} = 1$, we assume that both the monomorphic population of bidders and the $p$-mutant bid according to some affine linear functions of their respective values, namely $qz + c$ (with $(q, c) \in (0, 1)$ and $c = 1 - q$) and

$pv + d$ (with $(p, d) \in (0, 1)$ and $d = 1 - p$), respectively. Note that the analysis above required no common knowledge of $F$, whereas we now suppose that bidders know $\bar{v}$. The expected payoff of the $p$-mutant is

$$\Pi_S(p) = \int_{pv+1-p>qx+1-q} \int_{pv+1-p>qy+1-q} [v - \tfrac{n-1}{n}(qx + 1 - q)]\varphi(x, y)\ \mathrm{d}y\ \mathrm{d}x$$
$$+ \int_{pv+1-p<qx+1-q} \int_{pv+1-p>qy+1-q} \tfrac{1}{n}(pv + 1 - p)\varphi(x, y)\ \mathrm{d}y\ \mathrm{d}x$$
$$+ \int_{pv+1-p<qx+1-q} \int_{\substack{pv+1-p<qy+1-q \\ y<x}} \tfrac{1}{n}(qy + 1 - q)\varphi(x, y)\ \mathrm{d}y\ \mathrm{d}x.$$

or equivalently:

$$\Pi_S(p) = \int_0^{1+\frac{p(v-1)}{q}} \int_0^{1+\frac{p(v-1)}{q}} [v - \tfrac{n-1}{n}(qx + 1 - q)]\varphi(x, y)\ \mathrm{d}y\ \mathrm{d}x$$
$$+ \int_{1+\frac{p(v-1)}{q}}^1 \int_0^{1+\frac{p(v-1)}{q}} \tfrac{1}{n}(pv + 1 - p)\varphi(x, y)\ \mathrm{d}y\ \mathrm{d}x$$
$$+ \int_{1+\frac{p(v-1)}{q}}^1 \int_{1+\frac{p(v-1)}{q}}^x \tfrac{1}{n}(qy + 1 - q)\varphi(x, y)\ \mathrm{d}y\ \mathrm{d}x.$$

Taking the first-order condition with respect to $p$ and setting $p = q$ (the first condition for $q$ to be a best-reply to itself) yields:

$$\tfrac{1}{nq}(n-1)(n-2)(qv+1-q)(1-v)F(v)^{n-3} f(v)[1-F(v)]$$
$$+\tfrac{1}{n}(n-1)(v-1)F(v)^{n-2}[1-F(v)] + \tfrac{1}{nq}(n-1)(n-2)(qv+1-q)(v-1)F(v)^{n-3} f(v)[1-F(v)]$$
$$-\tfrac{(v-1)}{nq}(n-1)[1+q(v-1)]f(v)F(v)^{n-2} - \tfrac{1}{nq}(n-1)(v-1)[q-1+n(q-1)(v-1)-qv]F(v)^{n-2} f(v) = 0$$

whose solution, $q^*_S$, happens to be the slope of the equilibrium bidding strategy $b_S(v_i)$ when $F$ is uniform on $(0, 1)$, or equivalently a Beta distribution with $\alpha = \beta = 1$. In that case, the equilibrium strategy requires $q^*_S = n/(n + 1)$ and $c = 1/(n + 1)$, see Güth and Van Damme (1986).

# 4 Conclusion

By applying the indirect evolutionary approach (see Berninghaus et al., 2012), we justify risk neutral equilibrium bidding and extend the analysis of Güth and Pezanis-Christou (2015) who focus on auctions and could neglect the second-price auction format. When expecting $q_i v_i$ as the highest bid of $i$'s competitors, bidding $b_i = q_i v_i$ is optimal according to the first-price rule when $q_i < 1$ (as for auctions, Güth and Pezanis-Christou, 2015). Similarly, $q_i \geq 1$ can be rendered optimal for second-price fair division. So what our approach questions is not optimality *per se* but only the rationality of beliefs on which it is based. Whereas for second-price auctions conjectural belief evolution is obvious, in more restricted fair division bidding both pricing rules had to be considered. But even for this more restricted procedurally fair bidding, risk neutral bidding can be justified without presupposing the usual common knowledge assumption and counterfactual bidding.

Our analysis maintains some restrictive assumptions like independent private values and a priori symmetry of risk neutral bidders. The latter is evolutionarily captured by an infinite population and randomly formed bidder groups and fitness measured by average payoffs. In our view, the obvious next step would be to allow for a priori asymmetry in bidding contests (as in de Frutos, 2000). Altogether we have extended our *as-if* justification of rational bidding by *(i)* relying on psychologically valid conjectural beliefs and *(ii)* allowing for adaptation either, as in our studies, by evolutionary selection or by learning from past experiences via feedback information. In this way, we altogether justify rational bidding by adaptation rather than by cognitively demanding equilibrium analysis.[4]

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